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Generalized covariations, local time and Stratonovich Itô's formula for fractional Brownian motion with Hurst index $H \geq \frac{1}{4}$

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Abstract: Given a locally bounded real function g, we examine the existence of a 4-covariation $[g(B^H), B^H, B^H, B^H]$, where B^H is a fractional Brownian motion with a Hurst index $H \geq \frac{1}{4}$. We provide two essential applications. First, we relate the 4-covariation to one expression involving the derivative of local time, in the case $H = \frac{1}{4}$, generalizing an identity of Bouleau-Yor type, well-known for the classical Brownian motion. A second application is an Itô's formula of Stratonovich type for $f(B^H)$. The main difficulty comes from the fact B^H has only a finite 4-variation.

Key words and phrases: Fractional Brownian motion, fourth variation, Itô's formula, local time.

AMS Math Classification: Primary: 60H05, 60H10, 60H20; Secondary: 60G15, 60G48.

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1 Introduction

The present paper is devoted to generalized covariation processes and an Itô's formula related to the fractional Brownian motion. Classical Itô's formula and classical covariations constitute the core of stochastic calculus with respect to semimartingales. Fractional Brownian motion, which in general is not a semimartingale, has been studied intensively in stochastic analysis and it is considered in many applications in hydrology, telecommunications, economics and finance. Finance is the most recent one in spite of the fact, that, according to [31] the general assumption of no arbitrage opportunity is violated. Interesting remarks have been recently done by [7] and [40].

Recall that a mean zero Gaussian process $X = B^H$ is a fractional Brownian motion with Hurst index $H \in]0, 1[$ if its covariance function is given by

$$K_H(s,t) = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |s-t|^{2H}),\ (s,t) \in \mathbb{R}^2. \tag{1.1}$$

An easy consequence of that property is that

$$\mathrm{E}(B_t^H - B_s^H)^2 = (t-s)^{2H}. \tag{1.2}$$

Before concentrating on this self-similar Gaussian process, we would like to make some general observations.

Calculus with respect to integrands which are not semimartingales is now twenty years old. A huge amount of papers have been produced, and it is impossible to list them here; however we are still not so close from having a truely efficient approach for applications.

The techniques for studying non-semimartingales integrators are essentially three:

- Pathwise and related techniques.
- Dirichlet forms.
- Anticipating techniques (Malliavin calculus, Skorohod integration and so on).

Pathwise type integrals are defined very often using discretization, as limit of Riemann sums: an interesting survey on the subject is a book of R.M. Dudley and R. Norvaisa ([14]). They emphasize on a big historical literature in the deterministic case. The first contribution in the stochastic framework has been provided by H. Föllmer ([18]) in 1981; through this significant and simply written contribution, the author wished to discuss integration with respect to a Dirichlet process X, that is to say a local martingale plus a zero quadratic variation (or sometimes zero energy) process. In the sequel this approach has been continued and performed by J. Bertoin [4].

Since 1991, F. Russo and P. Vallois [35] have developed a regularization procedure, whose philosophy is similar to the discretization. They introduced a forward (generalizing Itô), backward, symmetric (generalizing Stratonovich) stochastic integrals and a generalized quadratic variation. Their techniques are of pathwise nature, but they are not truely pathwise. They make large use of ucp (uniform convergence in probability) related topology. More recently, several papers have followed that strategy, see for instance [36], [37], [38], [41], [16]. One advantage of the regularization technique is that it allows to generalize directly the classical Itô integral. Our forward integral of an adapted square integrable process with respect to the classical Brownian motion, is exactly Itô's integral; the integral via discretization is a sort of Riemann integral and it does allow to define easily for instance a totally discontinuous function as the indicator of rational numbers on $[0, 1]$. However the theorems contained in this paper can be translated without any difficulty in the language of discretization.

The terminology "Dirichlet processes" is inspired by the theory of Dirichlet forms. Tools from that theory have been developed to understand such processes as integrators, see for instance [27], [28]. Dirichlet processes belong to the class of finite quadratic variation processes.

Even though Dirichlet processes generalize semimartingales, fractional Brownian motion is a finite quadratic variation process (even Dirichlet) if and only if the Hurst index is greater or equal to $\frac{1}{2}$. When $H = \frac{1}{2}$, one obtains the classical standard Brownian motion. If $H > \frac{1}{2}$ it is even a zero quadratic variation process. Moreover fractional Brownian motion is a semimartingale if and only if it is a classical Brownian motion.

The regularization, or discretization technique, for those and related processes have been performed by [15], [17], [22], [39], [43] and [44] in the case of zero quadratic variation, so $H > \frac{1}{2}$. Young [42] integral can be often used under this circumstance. This integral coincides with the forward (but also with the backward or symmetric) integral since the covariation between integrand and integrator is always zero.

As we will explain later, when the integrator has paths with finite p-variation for $p > 2$, there is no hope to make use of forward and backward integrals and the reference integral will be for us the symmetric integral which is a generalization of Stratonovich integral.

The following step was done by T.J. Lyons and coauthors, see [25, 26], who considered, through an absolutely pathwise approach based on Lévy stochastic area, integrators having p-variation for any $p > 1$, provided one could construct a canonical geometric rough path associated with the process. This construction was done in [8] when the integrator is a fractional Brownian motion with Hurst parameter $H > \frac{1}{4}$; in that case, paths are almost surely of finite p-variation for $p > 4$.

Using Russo-Vallois regularization techniques, [16] has considered a stochastic calculus and some ordinary SDEs with respect to integrators with finite p-variation when $p \leq 3$. This applies directly to the fractional Brownian motion case for $H \geq \frac{1}{3}$. A significant object introduced in [16] was the concept of n-covariation $[Y_1, \ldots, Y_n]$ of n processes $Y_1, \ldots, Y_n$.

Since fractional Brownian motion is a Gaussian process, it was natural to use Skorohod-Malliavin approach, which as we said, constitutes a powerful tool for the analysis of integrators which are not semimartingales. Using this approach, integration with respect to fractional Brownian motion, was attacked by L. Decreusefonds and A. S. Ustunel [11] and it was studied intensively, see [6], [1] and [2], even when the integrator is a more general Gaussian process. Malliavin-Skorohod techniques allow to treat integration with respect to processes, in several situations where the variation is larger than 2. In particular [2] includes the case of a fractional Brownian motion B^H such that $H > \frac{1}{4}$. The key tool there, is the Skorohod integral which can be related to the symmetric-Stratonovich integral, up to a trace term of some Malliavin derivative of the integrand. In the case of fractional Brownian motion, [2] discussed a Itô's formula for the Stratonovich integral when the Hurst index H is strictly greater than $\frac{1}{4}$.

Other significant and interesting references about stochastic calculus with fractional Brownian motion, especially for $H > \frac{1}{2}$, are [12, 13, 24, 29, 30]. Some activity is also going on with stochastic PDE's driven by fractional sheets, see [21].

Our paper follows "almost pathwise calculus techniques" developed by Russo and Vallois, and it reaches the $H = \frac{1}{4}$ barrier, developing very detailed Gaussian calculations. As we said, one motivation of this paper, was to prove a Itô-Stratonovich formula for the fractional Brownian motion $X = B^H$ for $H \geq \frac{1}{4}$. Such a process has a finite 4-variation in the sense of [16] and a finite pathwise p-variation for $p > 4$, if one refers for instance to [14, 25]. We even prove that the cubic variation in the sense of [16] is zero even when the Hurst index is strictly bigger than $\frac{1}{6}$, see Proposition 2.3.

If one wants to remain in the framework of "pathwise" calculus, Itô's formula has to be of Stratonovich type. In fact, if $H < \frac{1}{2}$, such a formula cannot make use of the forward integral $\int_0^{\cdot} g(B^H) d^- B^H$ considered for instance in [36] because that integral, as well as the bracket $[g(B^H), B^H]$, is not defined since an explosion occurs in the regularization. For instance, as [2] points out, the forward integral $\int_0^T B_s^H d^- B_s^H$ does not exist. The use of Stratonovich-symmetric integral is natural and it provides cancellation of the term involving the second derivative.

Our Itô's formula is of the following type:

$$f(B_t^H) = f(B_0^H) + \int_0^t f'(B_u^H) d^\circ B_u^H.$$

As we said, when $H > \frac{1}{4}$, previous formula has already been treated by [2] using Malliavin calculus techniques.

The natural way to prove a Itô formula for an integrator having a finite 4-variation is to write a fourth order Taylor expansion:

$$f(X_{t+\varepsilon}) = f(X_t) + f'(X_t)(X_{t+\varepsilon} - X_t) + \frac{f''(X_t)}{2}(X_{t+\varepsilon} - X_t)^2$$

$$+\frac{f^{(3)}(X_t)}{6}(X_{t+\varepsilon} - X_t)^3 + \frac{f^{(4)}(X_t)}{24}(X_{t+\varepsilon} - X_t)^4$$

plus a remainder term which can be neglected. The second and third order terms can be essentially controlled because one will prove the existence of suitable covariations and the fourth order term provides a finite contribution because X has a finite fourth variation. If $H = \frac{1}{4}$, the third order term can be expressed in terms of a 4-covariation term $[f^{(3)}(X), X, X, X]$; it compensates then with the fourth order term.

At our point of view, the main achievement of this paper is the proof of the existence of the 4-covariation $[g(B^H), B^H, B^H, B^H]$, for $H \geq \frac{1}{4}$, g being locally bounded, see Theorem 3.7. Moreover, we prove that it is Hölder continuous with parameter strictly smaller than $\frac{1}{4}$. The local boundedness assumption on g can be of course relaxed, making a more careful analysis on the density of fractional Brownian motion at each instant. For the moment, we have not investigated that generality.

That result provides, as an application, the Itô-Stratonovich formula for $f(B^H)$, f being of class C^4, see Theorem 4.1.

A second application is a generalized Bouleau-Yor formula for fractional Brownian motion. Fractional Brownian motion B^H has a local time $(\ell_t^H(a))$ which has a continuous version in (a,t), for any $0 < H < 1$, as the density of the occupation measure, see for instance [3, 20]. In particular, one has

$$\int_0^t g(B_s^H)ds = \int_{\mathbb{R}} g(a)\ell_t^H(a)da.$$

First we mention the result for the classical Brownian motion $B = B^{\frac{1}{2}}$. A direct consequence of [19, 38] and [5] is the following: for a locally bounded function f , we have the equality,

$$[f(B), B]_t = -\int_{\mathbb{R}} f(a)\ell_t^{\frac{1}{2}}(da),$$

where the right hand side member is well-defined, since $(\ell_t^{\frac{1}{2}}(a))_{a\in\mathbb{R}}$ is a semimartingale. We will refer to the previous equality as to the *Bouleau-Yor* identity.

Our generalization of Bouleau-Yor identity is the following:

$$[f(B^{\frac{1}{4}}), B^{\frac{1}{4}}, B^{\frac{1}{4}}, B^{\frac{1}{4}}]_t = -3\int_{\mathbb{R}} f(a)(\ell_t^{\frac{1}{4}})'(a)da.$$

This is done in Corollary 3.8. We recall also that, for $H > \frac{1}{3}$, a Tanaka type formula has been obtained by [9] involving Skorohod integral.

The technique used here is a "pedestrian" but accurate exploitation of the Gaussian feature of fractional Brownian motion. Other recent papers where similar techniques have been used are for instance by [23] and [32]. Some of the computations are made using a Maple procedure.

The natural following question is the following: is $H = \frac{1}{4}$ an absolute barrier for the validity of Bouleau-Yor identity and for the Itô-Stratonovich pathwise formula?

Concerning the extended Bouleau-Yor identity, this is certainly not the case. Similar methods with more technicalities allow to establish the $2n-$ covariation $[g(B^H), B^H, \cdots, B^H]$ and its relation with the local time of B^H when $H = \frac{1}{2n}, n \geq 3$. We have decided to not develop these details because of the heavy technicalities.

As far as the "pathwise" Itô formula is concerned, it is a different story. It is of course immediate to see that for any $0 < H < 1$, if $B = B^H$, one has $B_t^2 = 2\int_0^t B_s d^\circ B_s$. On the other hand, proceeding by an obvious Taylor expansion, on would expect

$$B_t^3 = 3\int_0^t B_s^2 d^\circ B_s - \frac{1}{2}[B, B, B]_t \tag{1.3}$$

provided that $[B, B, B]_t$ exists; now Remark 2.4 below says that for $H < \frac{1}{6}$ this quantity does not exist and for $H > \frac{1}{6}$ it is zero. Therefore a Itô formula of the type (1.3) is valid for $H > \frac{1}{6}$ not valid for $H < \frac{1}{6}$. The study of a pathwise Itô formula for $H \in]\frac{1}{4}, \frac{1}{6}]$ is under our investigation.

The paper is organised as follows: we recall some basic definitions and results in section 2. In section 3 we state the theorems, we make some basic remarks and we prove part of the results. Section 4 is devoted to the proof of Itô's formula and section 5 contains the technical proofs.

2 Notations and recalls of preliminary results

We start by recalling some definitions and results established on some previous papers (see [36, 37, 38, 39]). In the following X and Y will be continuous processes. The space of continuous processes will be a metrizable Fréchet space $\mathcal{C}$, if it is endowed with the topology of the *uniform convergence in probability on each compact interval* (ucp). The space of random variables is also a metrizable Fréchet space, denoted by $\mathrm{L}^0(\Omega)$ and it is equipped with the topology of the convergence in probability.

We define the *forward integral*

$$\int_0^t Y_u d^- X_u := \lim_{\varepsilon \downarrow 0} \mathrm{ucp} \frac{1}{\varepsilon} \int_0^t Y_u (X_{u+\varepsilon} - X_u) du \tag{2.1}$$

and the *covariation*

$$[X, Y]_t := \lim_{\varepsilon \downarrow 0} \mathrm{ucp} \frac{1}{\varepsilon} \int_0^t (X_{u+\varepsilon} - X_u)(Y_{u+\varepsilon} - Y_u) du. \tag{2.2}$$

The *symmetric-Stratonovich integral* is defined as

$$\int_0^t Y_u d^\circ X_u := \lim_{\varepsilon \downarrow 0} \mathrm{ucp} \frac{1}{2\varepsilon} \int_0^t Y_u (X_{u+\varepsilon} - X_{(u-\varepsilon)\vee 0}) du \tag{2.3}$$

and the following fundamental equality is valid

$$\int_0^t Y_u d^\circ X_u = \int_0^t Y_u d^- X_u + \frac{1}{2}[X, Y]_t, \tag{2.4}$$

provided that the right member is well defined. However, as we will see in the next section, the left member may exist even if the covariation $[X, Y]$ does not exist. On the other hand the symmetric-Stratonovich integral can also be written as

$$\int_0^t Y_u d^\circ X_u = \lim_{\varepsilon \downarrow 0} \text{ucp} \int_0^t (Y_{u+\varepsilon} + Y_u) \frac{X_{u+\varepsilon} - X_u}{2\varepsilon} du. \tag{2.5}$$

Previous definitions will be somehow relaxed later.

If X is such that $[X, X]$ exists, X is called *finite quadratic variation process.* If $[X, X] = 0$, then X will be called *zero quadratic variation process.* In particular a *Dirichlet process* (the sum of a local martingale and a zero quadratic variation process) is a finite quadratic variation process. If X is finite quadratic variation process and if $f \in C^2(\mathbb{R})$, then the following Itô's formula holds:

$$f(X_t) = f(X_0) + \int_0^t f'(X_u) d^- X_u + \frac{1}{2} [f'(X), X]_t. \tag{2.6}$$

We recall that finite quadratic variation processes are stable by C^1 transformations. In particular, if $f, g \in \mathrm{C}^1$ and the vector (X, Y) is such that all mutual covariation exist, then $[f(X), g(Y)]_t = \int_0^t f'(X_s) g'(X_s) d[X, Y]_s$. Hence, formulas (2.4) and (2.6) give:

$$f(X_t) = f(X_0) + \int_0^t f'(X_u) d^\circ X_u. \tag{2.7}$$

Remark 2.1 *1. If X is a continuous semimartingale and Y is a suitable previsible process, then $\int_0^\cdot Y_u d^- X_u$ is the classical Itô's integral (for details see [36]).*

2. If X and Y are (continuous) semimartingales then $\int_0^\cdot Y_u d^\circ X_u$ is the Fisk-Stratonovich integral and $[X, Y]$ is the ordinary square bracket.

3. If $X = B^H$, then its paths are a.s. Hölder continuous with parameter strictly less than H. Therefore it is easy to see that, if $H > \frac{1}{2}$, then B^H is a zero quadratic variation process. When $H = \frac{1}{2}$, $B = B^{\frac{1}{2}}$ is the classical Brownian motion and so $[B^{\frac{1}{2}}, B^{\frac{1}{2}}]_t = t$. In particular Itô's formula (2.7) holds for $H \geq \frac{1}{2}$.

4. If $X = B$ is a classical Brownian motion, then formula (2.6) holds even for $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{R})$ (see [19, 38]). On the other hand, if $(\ell_t(a))$ is the local time associated with B, then in [5] it has shown that

$$f(B_t) = f(B_0) + \int_0^t f'(B_u) dB_u - \frac{1}{2} \int_{\mathbb{R}} f'(a) \ell_t(da). \tag{2.8}$$

The integral involving local time in the right member of (2.8) was defined directly by Bouleau and Yor, for a general semimartingale. However, in the case of Brownian motion, Corollary 1.13 in [5] states that for fixed $t > 0$, $(\ell_t(a))_{a \in \mathbb{R}}$ is a classical semimartingale; indeed that integral has a meaning as a deterministic Itô's integral. Thus, for $g \in L^2_{\mathrm{loc}}(\mathbb{R})$, setting f such that $f' = g$ and using (2.6) and (2.8), we obtain what will be called the **Bouleau-Yor identity***:*

$$\int_{\mathbb{R}} g(a) \ell_t(da) = -[g(B), B]_t. \tag{2.9}$$

Corollary 3.8 will generalize this result to the case of fractional Brownian motion $B^{\frac{1}{4}}$.

5. *An accurate study of "pathwise stochastic calculus" for finite quadratic variation processes has been done in [39]. One provides necessary and sufficient conditions on the covariance of a Gaussian process X so that X is a finite quadratic variation process and that X has a deterministic quadratic variation.*

Since the quadratic variation is not defined for B^H when $H < \frac{1}{2}$, we need to find a substitution tool. A concept of α-variation was already introduced in [39]. Here it will be called *strong α-variation* and is the following increasing continuous process:

$$[X]_t^{(\alpha)} := \lim_{\varepsilon\downarrow 0} \text{ucp} \int_0^t \frac{|X_{u+\varepsilon} - X_u|^\alpha}{\varepsilon} du. \tag{2.10}$$

A real attempt to adapt previous approach to integrators X which are not of finite quadratic variation has been done in [16]. For a positive integer n, in [16] one defines the *n-covariation* $[X^1, \dots, X^n]$ of a vector $(X^1, \dots, X^n)$ of real continuous processes, in the following way:

$$[X_1, \dots, X_n]_t := \lim_{\varepsilon\downarrow 0} \text{ucp} \int_0^t \frac{(X^1_{u+\varepsilon} - X^1_u)\dots(X^n_{u+\varepsilon} - X^n_u)}{\varepsilon} du. \tag{2.11}$$

Clearly, if $n = 2$, the 2-covariation $[X_1, X_2]$ is the covariation previously defined. In particular, if all the processes X_i are equal to X than the definition gives:

$$\underbrace{[X, \dots, X]}_{n\,\text{times}}(t) := \lim_{\varepsilon\downarrow 0} \text{ucp} \int_0^t \frac{(X_{u+\varepsilon} - X_u)^n}{\varepsilon} du, \tag{2.12}$$

which is called the *n-variation* of process X. Clearly, for even integer n, $[X]^{(n)} = \underbrace{[X, \dots, X]}_{n\,\text{times}}$.

Remark 2.2 1. *If the strong n-variation of X exists, then for all $m > n$, $\underbrace{[X, \dots, X]}_{m\,\text{times}} = 0$ (see [16], Remark 2.6.3, p. 7).*

2. *If $\underbrace{[X, \dots, X]}_{n\,\text{times}}$ and $[X]^{(n)}$ exist then, for $g \in C(\mathbb{R})$,*

$$\lim_{\varepsilon\downarrow 0} \text{ucp} \int_0^t g(X_u) \frac{(X_{u+\varepsilon} - X_u)^n}{\varepsilon} du = \int_0^t g(X_u) d[X, X, \dots, X]_u, \tag{2.13}$$

see [16], Remark 2.6.6, p. 8 and Remark 2.1, p. 5).

3. *Let $f_1, \dots, f_n \in C^1(\mathbb{R})$ and let X be a strong n-variation continuous process. Then*

$$[f_1(X), \dots, f_n(X)]_t = \int_0^t f_1'(X_u) \dots f_n'(X_u) d\underbrace{[X, \dots, X]}_{n\,\text{times}}(u).$$

4. *In [16], Proposition 3.4 one writes a Itô's type formula for X a continuous strong 3-variation process and for $f \in C^3(\mathbb{R})$:*

$$f(X_t) = f(X_0) + \int_0^t f'(X_u) d^\circ X_u - \frac{1}{12} \int_0^t f^{(3)}(X_u) d[X, X, X]_u. \tag{2.14}$$

In particular the previous point implies that

$$f(X_t) = f(X_0) + \int_0^t f'(X_u) d^\circ X_u - \frac{1}{12}[f''(X), X, X]_t.$$

5. *Let us come back to the process $X = B^H$. In [16], Proposition 3.1, it is proved that its strong 3-variation exists if $H \geq \frac{1}{3}$ but, even for the limiting case $H = \frac{1}{3}$, we have that the 3-covariation $[B^H, B^H, B^H] \equiv 0$.*

6. *In [39], Proposition 3.14, p. 22, it is proved that the strong $\frac{1}{H}$-variation of B^H exists and equals $\rho_H t$, where $\rho_H = \mathrm{E}[|G|^{\frac{1}{H}}]$, with G a standard normal random variable. Consequently,*

$$[B^H]^{(4)}_t = \begin{cases} 3t, & \text{if } H = \frac{1}{4} \\ 0, & \text{if } H > \frac{1}{4}. \end{cases} \tag{2.15}$$

In section 4, we will be able to write a Itô's formula for the fractional Brownian motion with index $\frac{1}{4} \leq H < \frac{1}{3}$. Let us stress that, in that case, B^H admits a (strong) 4-variation but not a strong 3-variation.

We end this section with the following remark: as it follows from the fifth part of the remark above, the 3-variation of a fractional Brownian motion B^H is zero when $H \geq \frac{1}{3}$. This result can be extended to the case of lower Hurst index:

Proposition 2.3 *Assume $H > \frac{1}{6}$. Then the 3-covariation $[B^H, B^H, B^H]$ exists and vanishes.*

Proof. For simplicity we fix $t = 1$. It suffices to prove that the limit when ε goes to zero of $\mathrm{E}[(\int_0^1 \frac{1}{\varepsilon}(B^H_{u+\varepsilon} - B^H_u)^3)^2]$, is zero. We will prove in fact that the limit, when $\varepsilon \downarrow 0$ of the following integral

$$\mathcal{I}_\varepsilon := 2 \iint_{0<u<v<1} \mathrm{E}\left(\frac{(B^H_{u+\varepsilon} - B^H_u)^3 (B^H_{v+\varepsilon} - B^H_v)^3}{\varepsilon^2} \right) dudv$$

equals zero.

For any centered Gaussian random vector (N, N') we have:

$$\mathrm{E}\left(N^3 (N')^3\right) = 6\mathrm{Cov}^3(N, N') + 9\mathrm{Cov}(N, N')\mathrm{Var}(N)\mathrm{Var}(N').$$

Indeed, it is enough to write $\mathrm{E}\left(N^3 (N')^3\right) = \mathrm{E}\left[N^3\, \mathrm{E}\left((N')^3 \mid N\right)\right]$ and to use linear regression (see also the proof of Lemma 3.7, p. 15 in [39] for a similar computation).

Denote $(N, N') = (B^H_{u+\varepsilon} - B^H_u, B^H_{v+\varepsilon} - B^H_v)$ and $\eta_\varepsilon(u, v) = \mathrm{Cov}(N, N')$. Therefore, previous integral $\mathcal{I}_\varepsilon$ can be written as

$$\mathcal{I}_\varepsilon = 12 \iint_{0<u<v<1} \frac{(\eta_\varepsilon(u, v))^3}{\varepsilon^2} dudv + 9 \cdot 2^{4H+1} \varepsilon^{4H-2} \iint_{0<u<v<1} \eta_\varepsilon(u, v) dudv =: \mathcal{I}^1_\varepsilon + \mathcal{I}^2_\varepsilon.$$

Since

$$\eta_\varepsilon(u, v) = \frac{1}{2}\left(|v - u + \varepsilon|^{2H} + |v - u - \varepsilon|^{2H} - 2|v - u|^{2H}\right),$$

a direct computation shows that

$$\int_0^v \eta_\varepsilon(u,v)du = \frac{1}{2(2H+1)} \begin{cases} (v+\varepsilon)^{2H+1} + (v-\varepsilon)^{2H+1} - 2v^{2H+1}, & \text{if } v \geq \varepsilon \\ (v+\varepsilon)^{2H+1} - (\varepsilon - v)^{2H+1} - 2v^{2H+1}, & \text{if } 0 \leq v \leq \varepsilon, \end{cases}$$

and then,

$$\iint_{0<u<v<1} \eta_\varepsilon(u,v)dudv = \int_0^\varepsilon dv \int_0^v \eta_\varepsilon(u,v)du + \int_\varepsilon^1 dv \int_0^v \eta_\varepsilon(u,v)du$$

$$\sim \frac{1}{H}\varepsilon^2 - \frac{1}{2H(H+1)(2H+1)}\varepsilon^{2H+2} \sim \frac{1}{H}\varepsilon^2, \text{ as } \varepsilon \downarrow 0.$$

Hence, $\mathcal{I}_\varepsilon^2 \sim 9 \cdot 2^{4H+1}\frac{1}{H}\varepsilon^{4H}$, when $\varepsilon \downarrow 0$, for any $H > 0$, and $\lim_{\varepsilon\downarrow 0} \mathcal{I}_\varepsilon^2 = 0$ for any $H > 0$.

To compute $\mathcal{I}_\varepsilon^1$ we set $\zeta = v - u$. Then

$$\mathcal{I}_\varepsilon^1 = \frac{3}{2\varepsilon^2}\int_0^1 \left((\zeta+\varepsilon)^{2H} + |\zeta - \varepsilon|^{2H} - 2\zeta^{2H}\right)^3 (1-\zeta)d\zeta$$

$$= 3\varepsilon^{6H-1}\int_0^{1/\varepsilon} \left((\theta+1)^{2H} + |\theta-1|^{2H} - 2\theta^{2H}\right)^3 (1-\varepsilon\theta)d\theta =: 3\varepsilon^{6H-1}\mathcal{I}_\varepsilon^{11} - 3\varepsilon^{6H}\mathcal{I}_\varepsilon^{12}.$$

Clearly, $\lim_{\varepsilon\downarrow 0}\mathcal{I}_\varepsilon^{11} = \int_0^\infty \left((\theta+1)^{2H} + |\theta-1|^{2H} - 2\theta^{2H}\right)^3 d\theta < \infty$, if $H < \frac{5}{6}$. A similar calculation shows that the second term tends to a convergent integral under the same condition on H. This yields $\mathcal{I}_\varepsilon^2 \sim 3\varepsilon^{6H-1}\int_0^\infty \left((\theta+1)^{2H} + |\theta-1|^{2H} - 2\theta^{2H}\right)^3 d\theta$, as $\varepsilon \downarrow 0$ and gives the conclusion, since $H > \frac{1}{6}$. ■

Remark 2.4 *¿From previous proof, we can also deduce that*

$$\lim_{\varepsilon\downarrow 0} \mathrm{E}\left[\left(\int_0^1 \frac{1}{\varepsilon}(B_{u+\varepsilon}^H - B_u^H)^3\right)^2\right]$$

is infinite for $H < \frac{1}{6}$; therefore if $H < \frac{1}{6}$, then 3-variation $[B^H, B^H, B^H]$ virtually does not exist.

3 Third order type integrals and 4-covariations

In order to understand the case of fractional Brownian motion for $H \geq \frac{1}{4}$, besides the family of integrals introduced until now, we need to introduce a new class of integrals.

Let again X, Y be continuous processes. We define the following **third order integrals** as follows: for $t > 0$,

$$\begin{aligned} \int_0^t Y_u d^{-3}X_u &:= \lim_{\varepsilon\downarrow 0}\text{prob}\,\frac{1}{\varepsilon}\int_0^t Y_u(X_{u+\varepsilon} - X_u)^3 du, \\ \int_0^t Y_u d^{+3}X_u &:= \lim_{\varepsilon\downarrow 0}\text{prob}\,\frac{1}{\varepsilon}\int_0^t Y_u(X_u - X_{(u-\varepsilon)\vee 0})^3 du, \\ \int_0^t Y_u d^{\circ 3}X_u &:= \lim_{\varepsilon\downarrow 0}\text{prob}\,\frac{1}{2\varepsilon}\int_0^t (Y_u + Y_{u+\varepsilon})(X_{u+\varepsilon} - X_u)^3 du. \end{aligned} \tag{3.1}$$

We will call them respectively (definite) **forward, backward** and **symmetric third order integral**. If the above $\mathrm{L}^0(\Omega)$-valued function,

$$t \mapsto \int_0^t Y_u d^{-3} X_u \quad \text{respectively} \quad t \mapsto \int_0^t Y_u d^{+3} X_u, \ t \mapsto \int_0^t Y_u d^{\circ 3} X_u$$

exists for any $t > 0$ (and equals 0 for $t = 0$), and it admits a continuous version, then such a version will be called **third order forward** (respectively **backward, symmetric**) **integral** and it will be denoted again by

$$\left(\int_0^t Y_u d^{-3} X_u\right)_{t \geq 0} \quad \text{respectively} \quad \left(\int_0^t Y_u d^{+3} X_u\right)_{t \geq 0}, \ \left(\int_0^t Y_u d^{\circ 3} X_u\right)_{t \geq 0}.$$

Remark 3.1 *If X is a strong 3-variation process, then $[X, X, X]$ will be a finite variation process and*

$$\int_0^t Y_u d^{-3} X_u = \int_0^t Y_u d^{+3} X_u = \int_0^t Y_u d[X, X, X]_u. \tag{3.2}$$

In particular, if $X = B^H$ is a fractional Brownian motion, with $H \geq \frac{1}{3}$, all the quantities in (3.2) are zero. If $H < \frac{1}{3}$ the strong 3-variation does not exists (see [16], Proposition 3). Recall that if $\frac{1}{6} < H < \frac{1}{3}$, the 3-covariation $[B^H, B^H, B^H]$ exists and vanishes (see Proposition 2.3), hence $\int_0^t Y_u d[X, X, X]_u = 0$. We shall prove that if $\frac{1}{4} < H < \frac{1}{3}$ and if $Y = g(B^H)$ then the third order integrals also vanish, so (3.2) is still true (see Theorem 3.4 below). If $H = \frac{1}{4}$ and $Y = g(B^H)$ the third order integrals are not necessarily zero.

The following results relate third order integrals with the notion of 4-covariation.

Proposition 3.2 *1.*

$$\int_0^t Y_u d^{\circ 3} X_u = \frac{1}{2}\left(\int_0^t Y_u d^{-3} X_u + \int_0^t Y_u d^{+3} X_u\right),$$

provided two of the three previous quantities exist.

2.

$$\int_0^t Y_u d^{+3} X_u - \int_0^t Y_u d^{-3} X_u = [Y, X, X, X]_t,$$

provided two of the three previous quantities exist.

Corollary 3.3 *Let X be a continuous process having a 4-variation and take $f \in C^1(\mathbb{R})$.*

1. *If $\int_0^t f(X_u) d^{-3} X_u$ exists, then $\int_0^t f(X_u) d^{+3} X_u$ exists and*

$$\int_0^t f(X_u) d^{+3} X_u = \int_0^t f(X_u) d^{-3} X_u + \int_0^t f'(X_u) d[X, X, X, X]_u.$$

2. *If $\int_0^t f'(X_u) d^{-3} X_u$ exists and if furthermore $f \in C^2(\mathbb{R})$, then*

$$[f(X), X, X]_t = \int_0^t f'(X_u) d^{-3} X_u + \frac{1}{2} \int_0^t f''(X_u) d[X, X, X, X]_u.$$

Proof. The first point follows immediately from Proposition 3.2 and Remark 2.2 3). To prove the second part, a second order Taylor expansion gives, for $s, \varepsilon > 0$,

$$f(X_{s+\varepsilon}) - f(X_s) = f'(X_s)(X_{s+\varepsilon} - X_s) + \frac{f''(X_s)}{2}(X_{s+\varepsilon} - X_s)^2 + R(f, \varepsilon, s)(X_{s+\varepsilon} - X_s)^2,$$

where $R(f, \varepsilon, s)$ converges to zero, ucp in s, when ε goes to zero, by the uniform continuity of f and of paths of X on each compact interval. Multiplying the previous expression by $(X_{s+\varepsilon} - X_s)^2$, integrating from 0 to t, dividing by ε and using Remark 2.2 2) we obtain the result. ■

In spite of the now classical notion of the symmetric integral given in (2.5), we need to relax this definition. ¿From now on, we will say that the symmetric integral of a process Y with respect to an integrator X if

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t Y_u (X_{u+\varepsilon} - X_{(u-\varepsilon)\vee 0}) du$$

exists *in probability* and the limiting $\mathrm{L}^0(\Omega)$-valued function has a continuous version. We will still denote that process (unique up to indistinguishability) by $\int_0^t Y_u d^\circ X_u$.

Similarly, in this paper the concept of 4-covariation will be understood in a weaker sense with respect to (2.11).

We will say that the 4-covariation $[X^1, X^2, X^3, X^4]$ exists if

$$\lim_{\varepsilon \downarrow 0} \int_0^t \frac{(X^1_{u+\varepsilon} - X^1_u) \dots (X^4_{u+\varepsilon} - X^4_u)}{\varepsilon} du$$

exists in probability and if that the limiting $\mathrm{L}^0(\Omega)$ valued function has a continuous version.

Clearly if $\int_0^t Y_u d^\circ X_u$ exists in the classical sense of Russo and Vallois, then it exists also in this relaxed meaning; similarly, if $[X^1, X^2, X^3, X^4]$ exists in the (2.11) sense, that it will exist in the relaxed sense. We remark that when all the processes are equal, then a Dini type lemma, as in [39] allows to show that the two definitions of 4-covariations are equivalent. We remark that Proposition 3.2 and Corollary 3.3 are still valid with these conventions.

¿From now on we will concentrate on the case when $X = B^H$ is the fractional Brownian motion with Hurst index H.

In the statement of the fundamental result of this section we use the following definition: we say that a real function g fulfills the *subexponential inequality* if

$$|g(x)| \le L e^{\ell |x|}, \text{ with } \ell, L \text{ positive constants.} \tag{3.3}$$

Theorem 3.4 *Let $\frac{1}{4} \le H < \frac{1}{3}$, $t > 0$, and g be a real locally bounded function. The following properties hold:*

a) The third order integrals $\int_0^t g(B^H_u) d^{\pm 3} B^H_u$ exist and vanish if $\frac{1}{4} < H < \frac{1}{3}$.

Henceforth, we assume $H = \frac{1}{4}$.

b) The third order integrals $\int_0^t g(B^{\frac{1}{4}}_u) d^{\pm 3} B^{\frac{1}{4}}_u$ exist and are opposite, that is for any $t > 0$

$$\int_0^t g(B^{\frac{1}{4}}_u) d^{+3} B^{\frac{1}{4}}_u = - \int_0^t g(B^{\frac{1}{4}}_u) d^{-3} B^{\frac{1}{4}}_u. \tag{3.4}$$

Moreover, the processes $\left(\int_0^t g(B_u^{\frac14})d^{\pm 3}B_u^{\frac14}\right)_{t\geq 0}$*, are Hölder continuous with parameter strictly less than* $\frac14$*.*

c) *If furthermore g fulfills the subexponential inequality (3.3), the expectation and the second moment of third order integrals are given by*

$$\mathrm{E}\left\{\int_0^t g(B_u^{\frac14})d^{-3}B_u^{\frac14}\right\} = -\mathrm{E}\left\{\int_0^t g(B_u^{\frac14})d^{+3}B_u^{\frac14}\right\} = -\frac{3}{2}\int_0^t \frac{du}{\sqrt{u}}\mathrm{E}[g(B_u^{\frac14})B_u^{\frac14}] \quad (3.5)$$

and

$$\mathrm{E}\left\{\left(\int_0^t g(B_u^{\frac14})d^{\pm 3}B_u^{\frac14}\right)^2\right\} = \frac{9}{2}\iint_{0<u<v<t} du\,dv\,\mathrm{E}\Big[g(B_u^{\frac14})g(B_v^{\frac14}) \quad (3.6)$$

$$\times\left(\lambda_{11}\lambda_{12}(B_u^{\frac14})^2 + (\lambda_{11}\lambda_{22}+\lambda_{12}^2)B_u^{\frac14}B_v^{\frac14} + \lambda_{12}\lambda_{22}(B_v^{\frac14})^2 - \lambda_{12}\right)\Big],$$

where the right hand sides of (3.5) and (3.6) are absolute convergent integrals. Here

$$\lambda_{11} = \frac{\sqrt{v}}{\sqrt{uv} - K_{1/4}(u,v)^2},\ \lambda_{22} = \frac{\sqrt{u}}{\sqrt{uv} - K_{1/4}(u,v)^2},\ \lambda_{12} = -\frac{K_{1/4}(u,v)}{\sqrt{uv} - K_{1/4}(u,v)^2}. \quad (3.7)$$

d) *If* $g \in C^1(\mathbb{R})$ *then the quantity in (3.4) is equal to* $\frac12\int_0^t g'(B_u^{\frac14})d[B^{\frac14}]_u^{(4)}$*.*

The proof of Theorem 3.4 is postponed to the last section. Let us note that composing Borel functions and fractional Brownian motion is authorised:

Remark 3.5 *If g is a, Lebesgue a.e. defined, locally bounded Borel function then the composition* $g(B_t^H)$*,* $t > 0$ *is a well defined, up to an a.s. equivalence, random variable. Precisely, if* g_1, g_2 *are two Lebesgue a.e. modifications of g then* $g_1(B_t^H) = g_2(B_t^H)$ *a.s. (since* B_t^H *has a density function). Consequently,* $\int_0^t g_1(B_u^H)d^{\pm 3}B_u^H$ *exists if and only if* $\int_0^t g_2(B_u^H)d^{\pm 3}B_u^H$ *exists and are equal.*

The proof of the following result is easy obtained by a localization argument:

Proposition 3.6 *The maps* $g \mapsto \int_0^t g(B_u^{\frac14})d^{\pm 3}B_u^{\frac14}$ *and* $g \mapsto \int_0^t g(B_u^{\frac14})d^{\circ 3}B_u^{\frac14}$ *are continuous from* $L^\infty_{\mathrm{loc}}(\mathbb{R})$ *to* $\mathrm{L}^0(\Omega)$*.*

Next result states the existence of a significant fourth order covariation related to the fractional Brownian motion B^H with Hurst index $H = \frac14$. Its proof is obvious using parts *b)* and *d)* in Theorem 3.4, Proposition 3.6, Proposition 3.2 2) and Remark 2.2 3).

Theorem 3.7 *Let* $g \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ *and fix* $t > 0$*. The process* $([g(B^{\frac14}), B^{\frac14}, B^{\frac14}, B^{\frac14}]_t)_{t\geq 0}$ *is well defined, has Hölder continuous paths of parameter strictly less than* $\frac14$ *and is given by:*

$$[g(B^{\frac14}), B^{\frac14}, B^{\frac14}, B^{\frac14}]_t = 2\int_0^t g(B_u^{\frac14})d^{+3}B_u^{\frac14} = -2\int_0^t g(B_u^{\frac14})d^{-3}B_u^{\frac14}. \quad (3.8)$$

One consequence of Theorem 3.7 concerns the local time of the fractional Brownian motion. Let $(\ell_t^H(a))$ be the local time as the occupation measure density (see [3, 20]). It exists for any $0 < H < 1$; moreover, if $H < \frac{1}{3}$, it is absolutely continuous with respect to a. We denote by $(\ell_t^H)'(a)$ the corresponding derivative. The following result extends to the fractional Brownian motion with $H = \frac{1}{4}$, the Bouleau-Yor type equality (2.9) discussed at Remark 2.1 for the case of the classical Brownian motion:

Corollary 3.8 *Let $g \in L^\infty_{\text{loc}}$. Then, for fixed $t > 0$,*

$$[g(B^{\frac{1}{4}}), B^{\frac{1}{4}}, B^{\frac{1}{4}}, B^{\frac{1}{4}}]_t = -3 \int g(a)(\ell_t^{\frac{1}{4}})'(a)da. \tag{3.9}$$

Proof. Recall that $[g(B^{\frac{1}{4}}), B^{\frac{1}{4}}, B^{\frac{1}{4}}, B^{\frac{1}{4}}]_t = 3t$ and so $[g(B^{\frac{1}{4}}), B^{\frac{1}{4}}, B^{\frac{1}{4}}, B^{\frac{1}{4}}]_t = 3\int_0^t g'(B_s^{\frac{1}{4}})ds$, whenever $g \in C^1(\mathbb{R})$ with compact support. By density occupation formula, previous expression becomes $-3\int g'(a)\ell_t^{\frac{1}{4}}(a)da$. Integrating by parts, we obtain the right member of (3.9). This shows the equality for smooth g. To obtain the final statement, we regularize $g \in L^\infty_{\text{loc}}(\mathbb{R})$ by taking $g_n = g * \phi_n$, where (ϕ_n) is a sequence of mollifiers converging to the Dirac delta function, we apply the equality for g being smooth and we take the limit. For the limit of left members, we use the continuity of the considered 4-covariation. For the right members, we use the Lebesgue dominated convergence theorem: in fact with recall that $a \to \lambda_t'(a)$ is integrable with compact support and on each compact the upper bound of $|g_n|$ is bounded by the upper bound of $|g|$. ■

4 Itô's formula

Let B^H be again a fractional Brownian motion with Hurst index H.

Theorem 4.1 *Let $H \geq \frac{1}{4}$ and $f \in C^4(\mathbb{R})$.*

Then the symmetric integral $\int_0^t f'(B_u^H)d^\circ B_u^H$ exists and a Itô's type formula can be written:

$$f(B_t^H) = f(B_0^H) + \int_0^t f'(B_u^H)d^\circ B_u^H. \tag{4.1}$$

Remark 4.2 *The most interesting case concerns the critical limiting case $H = \frac{1}{4}$. When, $H > \frac{1}{4}$ the result was also established in [2] using other methods.*

Proof. Theorem 4.1 will be a consequence of Theorem 3.4. Let fix $t > 0$. In fact, we prove that, for any $f \in C^4(\mathbb{R})$,

$$f(B_t^H) = f(B_0^H) + \int_0^t f'(B_u^H)d^\circ B_u^H - \frac{1}{12}\int_0^t f^{(3)}(B_u^H)d^{\circ 3}B_u^H, \tag{4.2}$$

which implies the final result since $\int_0^t f^{(3)}(B_s^H)d^{\circ 3}B_s^H$ vanishes (see Theorem 3.4 a),b) and Proposition 3.2 1)).

We start with Taylor formula: for $a, b \in \mathbb{R}$ we have

$$f(b) - f(a) = f'(a)(b-a) + f''(a)\frac{(b-a)^2}{2} + f^{(3)}(a)\frac{(b-a)^3}{6} \tag{4.3}$$

$$+\frac{(b-a)^4}{6}\int_0^1 \lambda^3 f^{(4)}\left(\lambda a+(1-\lambda)b\right)d\lambda$$

and also

$$f(a)-f(b)=f'(b)(a-b)+f''(b)\frac{(a-b)^2}{2}+f^{(3)}(b)\frac{(a-b)^3}{6}$$

$$+\frac{(a-b)^4}{6}\int_0^1 \lambda^3 f^{(4)}\left(\lambda b+(1-\lambda)a\right)d\lambda=-f'(b)(b-a)+f''(b)\frac{(b-a)^2}{2}-f^{(3)}(b)\frac{(b-a)^3}{6}$$

$$+\frac{(b-a)^4}{6}\int_0^1 (1-\lambda)^3\left(f^{(4)}(\lambda a+(1-\lambda)b)\right)d\lambda.$$

Since

$$f''(b)=f''(a)+f^{(3)}(a)(b-a)+(b-a)^2\int_0^1 \lambda\left(f^{(4)}(\lambda a+(1-\lambda)b)\right)d\lambda$$

and

$$f^{(3)}(b)=f^{(3)}(a)+(b-a)\int_0^1\left(f^{(4)}(\lambda a+(1-\lambda)b)\right)d\lambda,$$

we can write

$$f(a)-f(b)=-f'(b)(b-a)+f''(a)\frac{(b-a)^2}{2}+f^{(3)}(a)\frac{(b-a)^3}{3} \tag{4.4}$$

$$+(b-a)^4\int_0^1\left(\frac{\lambda^2}{2}-\frac{\lambda^3}{6}\right)f^{(4)}(\lambda a+(1-\lambda)b)d\lambda.$$

Taking the difference between (4.3) and (4.4) and dividing by 2, we get

$$f(b)-f(a)=\frac{f'(a)+f'(b)}{2}(b-a)-\frac{1}{12}f^{(3)}(a)(b-a)^3 \tag{4.5}$$

$$+(b-a)^4\int_0^1\left(\frac{\lambda^3}{6}-\frac{\lambda^2}{4}\right)f^{(4)}(\lambda a+(1-\lambda)b)d\lambda.$$

On the other hand, exchanging roles of a and b, we get

$$f(a)-f(b)=-\frac{f'(a)+f'(b)}{2}(b-a)+\frac{1}{12}f^{(3)}(b)(b-a)^3 \tag{4.6}$$

$$+(b-a)^4\int_0^1\left(\frac{(1-\lambda)^3}{6}-\frac{(1-\lambda)^2}{4}\right)f^{(4)}(\lambda a+(1-\lambda)b)d\lambda.$$

Taking this time the difference between (4.5) and (4.6) and dividing by 2, we obtain

$$f(b)-f(a)=\frac{f'(a)+f'(b)}{2}(b-a)-\frac{f^{(3)}(a)+f^{(3)}(b)}{24}(b-a)^3+(b-a)^4J(a,b), \tag{4.7}$$

where

$$J(a,b)=\int_0^1\left(\frac{\lambda^3}{6}-\frac{\lambda^2}{4}+\frac{1}{24}\right)f^{(4)}(\lambda a+(1-\lambda)b)d\lambda$$

$$=\int_0^1\left(\frac{\lambda^3}{6}-\frac{\lambda^2}{4}+\frac{1}{24}\right)\left(f^{(4)}(\lambda a+(1-\lambda)b)-f^{(4)}(a)\right)d\lambda,$$

since $\int_0^1 \left(\frac{\lambda^3}{6} - \frac{\lambda^2}{4} + \frac{1}{24}\right) d\lambda = 0$.
Setting in (4.7) $a = B_u^H$ and $b = B_{u+\varepsilon}^H$, we get

$$f(B_{u+\varepsilon}^H) - f(B_u^H) = \left(f'(B_u^H) + f'(B_{u+\varepsilon}^H)\right) \frac{B_{u+\varepsilon}^H - B_u^H}{2} \tag{4.8}$$
$$- \frac{f^{(3)}(B_u^H) + f^{(3)}(B_{u+\varepsilon}^H)}{2} \frac{(B_{u+\varepsilon}^H - B_u^H)^3}{12} + J(B_u^H, B_{u+\varepsilon}^H)(B_{u+\varepsilon}^H - B_u^H)^4.$$

Using the uniform continuity on each compact real interval I of $f^{(4)}$ and of B^H, we observe that $\sup_{u \in I} J(B_u^H, B_{u+\varepsilon}^H) \to 0$, in probability when $\varepsilon \downarrow 0$. Take $t > 0$, integrate (4.8) in u on $[0, t]$ and divide by ε:

$$\frac{1}{\varepsilon} \int_0^t (f(B_{u+\varepsilon}^H) - f(B_u^H))du = \int_0^t (f'(B_{u+\varepsilon}^H) + f'(B_u^H)) \frac{B_{u+\varepsilon}^H - B_u^H}{2\varepsilon} du$$
$$- \int_0^t \frac{f^{(3)}(B_u^H) + f^{(3)}(B_{u+\varepsilon}^H)}{2} \frac{(B_{u+\varepsilon}^H - B_u^H)^3}{12\varepsilon} du + \int_0^t J(B_u^H, B_{u+\varepsilon}^H) \frac{(B_{u+\varepsilon}^H - B_u^H)^4}{\varepsilon} du.$$

By a simple change of variable we can transform the left-hand side and we finally obtain

$$\frac{1}{\varepsilon} \int_t^{t+\varepsilon} f(B_u^H)du - \frac{1}{\varepsilon} \int_0^{\varepsilon} f(B_u^H)du = \int_0^t (f'(B_{u+\varepsilon}^H) + f'(B_u^H)) \frac{B_{u+\varepsilon}^H - B_u^H}{2\varepsilon} du \tag{4.9}$$
$$- \int_0^t \frac{f^{(3)}(B_u^H) + f^{(3)}(B_{u+\varepsilon}^H)}{2} \frac{(B_{u+\varepsilon}^H - B_u^H)^3}{12\varepsilon} du + \int_0^t J(B_u^H, B_{u+\varepsilon}^H) \frac{(B_{u+\varepsilon}^H - B_u^H)^4}{\varepsilon} du.$$

The left-hand side of (4.9) tends, as $\varepsilon \downarrow 0$, toward $f(B_t^H) - f(B_0^H)$. Since $\sup_{u \in [0,t]} J(B_u^H, B_{u+\varepsilon}^H)$ tends to zero, the last term on the right-hand side of (4.9) too tends to zero, by the existence of the strong 4-variation. The second term in the right-hand side converges to $\int_0^t f^{(3)}(B_u^H) d^{\circ 3} B_u^H$, which exists by Theorem 3.4. Therefore, the first term on the right-hand side of (4.9) is also forced to have a limit in probability. According to point *b)* of Theorem 3.4, the symmetric third order integral has a continuous version in t; therefore the second term must have a continuous version and it will be of course the symmetric integral $\int_0^t f'(B_u^H) d^{\circ} B_u^H$. (4.2) is proved.

■

5 Proofs of existence and properties of third order integrals

The main topic of this section is the proof of Theorem 3.4 which will be articulated from step *I)* to step *VI)*.

Recall that $\frac{1}{4} \leq H < \frac{1}{3}$. We will consider only the third order forward integral, since for the third order backward integral the reasoning is similar. Hence, let us denote

$$I_\varepsilon(g)(t) := \frac{1}{\varepsilon} \int_0^t g(B_u^H)(B_{u+\varepsilon}^H - B_u^H)^3 du, \tag{5.1}$$

and recall that the forward third order integral $\int_0^t g(B_u^H) d^{-3} B_u^H$ was defined as the limit in probability of $I_\varepsilon(g)(t)$. For simplicity we will fix $t = 1$ and simply denote $I_\varepsilon(g) := I_\varepsilon(g)(1)$.

First let us describe the plan of Theorem 3.4 proof.

I) Computation of $\lim_{\varepsilon\downarrow 0} \mathrm{E}[I_\varepsilon(g)]$. *The limit vanishes for* $\frac{1}{4} < H < \frac{1}{3}$. *If* $H = \frac{1}{4}$ *and assuming the existence stated in point b) the computation also gives (3.5).*

II) Computation of $\lim_{\varepsilon\downarrow 0} \mathrm{E}[I_\varepsilon(g)^2]$. *We state Lemma 5.1 which allows to give an equivalent of this second moment as* $\varepsilon \downarrow 0$. *Again the limit vanishes for* $\frac{1}{4} < H < \frac{1}{3}$, *hence we get point a). Henceforth we assume* $H = \frac{1}{4}$. *(3.6) is obtained assuming again the existence stated in b).*

III) Integrals on the right hand sides of (3.5) and (3.6) are absolute convergent and the proof of point c) is complete.

IV) Proof of the existence of the forward third order integral (as a first step in proving b)). First we reduce the study to the case of a bounded function g and then we establish the existence under this hypothesis.

V) We prove the existence of a continuous version of the forward third order integral and the Hölder regularity of its paths.

VI) End of point b) proof: we verify (3.4) proving at the same time d). We state and use Lemma 5.3.

The end of the section is devoted to the proofs of Lemmas 5.1 and 5.3 which are stated at steps *II), VI)* and used in the proof of points *b), d)* of Theorem 3.4.

I) Computation of $\lim_{\varepsilon\downarrow 0} \mathrm{E}[I_\varepsilon(g)]$.
To compute the expectation of $I_\varepsilon(g)$ we will use the linear regression for $B^H_{u+\varepsilon} - B^H_u$, which is a centered Gaussian random variable with variance ε^{2H}. It can be written as

$$B^H_{u+\varepsilon} - B^H_u = \frac{K_H(u, u+\varepsilon) - K_H(u,u)}{K_H(u,u)} B^H_u + Z_\varepsilon, \tag{5.2}$$

where Z_ε is a Gaussian mean-zero random variable, independent from B^H_u with variance $\varepsilon^{2H} - \frac{1}{4u^{2H}}((u+\varepsilon)^{2H} - u^{2H} - \varepsilon^{2H})^2$. Therefore,

$$B^H_{u+\varepsilon} - B^H_u = \alpha_\varepsilon(u) B^H_u + \beta_\varepsilon(u) N, \tag{5.3}$$

where N is a standard normal random variable independent from B^H_u and where, for $u > 0$ fixed, as $\varepsilon \downarrow 0$,

$$\alpha_\varepsilon(u) := \frac{1}{2u^{2H}} \left((u+\varepsilon)^{2H} - u^{2H} - \varepsilon^{2H}\right) = \frac{1}{2} \left(\frac{\varepsilon}{u}\right)^{2H} \phi_0\left(\frac{\varepsilon}{u}\right) \tag{5.4}$$

and

$$\beta^2_\varepsilon(u) := \varepsilon^{2H} - \alpha^2_\varepsilon(u) u^{2H} = \varepsilon^{2H} \phi_1\left(\frac{\varepsilon}{u}\right), \tag{5.5}$$

where $x^{2H}\phi_0(x) := (1+x)^{2H} - 1 - x^{2H}$, $\phi_1(x) := \left(1 - \frac{1}{4}x^{2H}\phi_0^2(x)\right)_+$, with ϕ_0 being a continuous bounded function, ϕ_1 a bounded function with the property $\lim_{x\downarrow 0}\phi_0(x) = -1$, $\lim_{x\downarrow 0}\phi_1(x) = 1$. Since $2H < 1$ we can also write

$$\alpha_\varepsilon(u) = -\frac{\varepsilon^{2H}}{2u^{2H}} \left(1 - 2Hu^{2H-1}\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right), \text{ as } \varepsilon \downarrow 0. \tag{5.6}$$

Moreover

$$\beta_\varepsilon^2(u) = \varepsilon^{2H}\left(1 - \frac{\varepsilon^{2H}}{4u^{2H}}\right) + o(\varepsilon^{4H}) \text{ as } \varepsilon \downarrow 0. \tag{5.7}$$

We can now compute the first moment of $I_\varepsilon(g)$. Replacing (5.3) in the expression of $I_\varepsilon(g)$ and from the independence of N and B_u^H, we obtain

$$\mathrm{E}\left[I_\varepsilon(g)\right] = \int_0^1 \frac{\alpha_\varepsilon^3(u)}{\varepsilon}\mathrm{E}\left[g(B_u^H)(B_u^H)^3\right] du + \int_0^1 \frac{3\alpha_\varepsilon(u)\beta_\varepsilon^2(u)}{\varepsilon}\mathrm{E}\left[g(B_u^H)B_u^H\right] du.$$

Cauchy-Schwarz inequality and the hypothesis on g imply that, for $0 < u < 1$,

$$\mathrm{E}\left[|g(B_u^H)B_u^H|\right] \le L\mathrm{E}\left[e^{\ell|B_u^H|}|B_u^H|\right] \le L'\mathrm{E}\left[e^{\ell B_u^H}|B_u^H|\right] \le \text{const.}\sqrt{\mathrm{E}[(B_u^H)^2]} \le \text{const.}u^H < \infty.$$

In a similar way, it follows

$$\mathrm{E}\left[|g(B_u^H)(B_u^H)^3|\right] \le \text{const.}\sqrt{\mathrm{E}[(B_u^H)^6]} = \text{const.}\, u^{3H}.$$

Hence, since $\frac{1}{4} \le H < \frac{1}{3}$, as $\varepsilon \downarrow 0$,

$$\frac{\alpha_\varepsilon^3(u)}{\varepsilon}u^{3H} = \frac{1}{8}\frac{\varepsilon^{6H-1}}{u^{3H}}\phi_0^3\left(\frac{\varepsilon}{u}\right), \text{ with } \int_0^1 \frac{\mathrm{du}}{\mathrm{u^{3H}}} < \infty.$$

Since $\frac{1}{4} \le H < \frac{1}{3}$, letting ε go to 0 we get

$$\lim_{\varepsilon\downarrow 0}\mathrm{E}\left[I_\varepsilon(g)\right] = \int_0^1 \left(\lim_{\varepsilon\downarrow 0}\frac{3\alpha_\varepsilon(u)\beta_\varepsilon^2(u)}{\varepsilon}\right)\mathrm{E}\left[g(B_u^H)B_u^H\right] du$$

and (3.5) is obtained using (5.4) and (5.5). Indeed, since $\frac{1}{4} \le H < \frac{1}{3}$, we have

$$\frac{\alpha_\varepsilon(u)\beta_\varepsilon^2(u)}{\varepsilon}u^H = \frac{1}{2}\frac{\varepsilon^{4H-1}}{u^H}(\phi_0\phi_1)\left(\frac{\varepsilon}{u}\right), \text{ with } \int_0^1 \frac{du}{u^H} < \infty.$$

Clearly,

$$\lim_{\varepsilon\downarrow 0}\mathrm{E}\left[I_\varepsilon(g)\right] = 0, \text{ if } \frac{1}{4} < H < \frac{1}{3}. \tag{5.8}$$

If $H = \frac{1}{4}$, Lebesgue dominated convergence implies that

$$\lim_{\varepsilon\downarrow 0}\mathrm{E}\left[I_\varepsilon(g)\right] = -\frac{3}{2}\int_0^1 \frac{1}{\sqrt{u}}\mathrm{E}\left[g(B_u^H)B_u^H\right] du$$

and then (3.5) follows assuming the existence in the first part of point *b)* of Theorem 3.4. Let us also explain the opposite sign in (3.5) for the backward third order integral. We need to consider (see (5.3)

$$B_u^H - B_{u-\varepsilon}^H = \hat{\alpha}_\varepsilon(u)B_u^H + \hat{\beta}_\varepsilon(u)N \text{ (assume that } u - \varepsilon > 0),$$

where (see (5.4) and (5.5))

$$\hat{\alpha}_\varepsilon(u) = \frac{1}{2u^{2H}}\left(u^{2H} - (u-\varepsilon)^{2H} + \varepsilon^{2H}\right), \; \hat{\beta}_\varepsilon(u)^2 = \varepsilon^{2H} - \hat{\alpha}_\varepsilon(u)^2 u^{2H}.$$

Hence (see (5.6))

$$\hat{\alpha}_\varepsilon(u) = \frac{\varepsilon^{2H}}{2u^{2H}}\left(1 + 2Hu^{2H-1}\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right), \text{ as } \varepsilon \downarrow 0,$$

while (5.7) is still true for $\hat{\beta}_\varepsilon(u)^2$. These relations give the opposite sign in (3.5) for the backward third order integral. □

II) Computation of $\lim_{\varepsilon\downarrow 0} \mathrm{E}[I_\varepsilon(g)^2]$.
The computation of the second moment of $I_\varepsilon(g)$ is done using again the Gaussian feature of the process. We express the linear regression for the random vector $(B^H_{u+\varepsilon} - B^H_u, B^H_{v+\varepsilon} - B^H_v)$. We denote $G = (G_1, G_2, G_3^\varepsilon, G_4^\varepsilon)$ the Gaussian mean-zero random vector $(B^H_u, B^H_v, B^H_{u+\varepsilon} - B^H_u, B^H_{v+\varepsilon} - B^H_v)$ and we use a similar idea as in *I)*. For instance (5.2) will be replaced by

$$\begin{pmatrix} G_3^\varepsilon \\ G_4^\varepsilon \end{pmatrix} = A_\varepsilon \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} + \begin{pmatrix} Z_1^\varepsilon \\ Z_2^\varepsilon \end{pmatrix}, \tag{5.9}$$

where the Gaussian mean-zero random vector $Z^\varepsilon = (Z_1^\varepsilon, Z_2^\varepsilon)$ is independent from (G_1, G_2). Clearly,

$$I_\varepsilon(g)^2 = 2\iint\limits_{0<u<v<1} g(B^H_u)g(B^H_v)\frac{(B^H_{u+\varepsilon} - B^H_u)^3}{\varepsilon}\frac{(B^H_{v+\varepsilon} - B^H_v)^3}{\varepsilon} dudv,$$

hence

$$\mathrm{E}\left[I_\varepsilon(g)^2\right] = 2\mathrm{E}\left\{\iint_{0<u<v<1} g(G_1)g(G_2)\,\mathrm{E}\left(\frac{(G_3^\varepsilon)^3(G_4^\varepsilon)^3}{\varepsilon^2} \mid G_1, G_2\right) dudv\right\}. \tag{5.10}$$

Therefore we need to compute the conditional expectation in (5.10). For that reason, we need the following lemma which will be useful again at step *IV 2)* where we prove the existence of the L^2-limit of I_ε. For random variables ξ, ζ, ϕ_ε, we will denote

$$\xi \overset{(\text{law})}{=} \zeta + o(\varepsilon) \text{ as } \varepsilon \downarrow 0, \text{ if } \xi \overset{(\text{law})}{=} \zeta + \varepsilon\phi_\varepsilon, \text{ with } \mathrm{E}\left[\sup_{0<\varepsilon<1} |\phi_\varepsilon|^p\right] < \infty, \forall p.$$

Lemma 5.1 *Consider the Gaussian mean-zero random vector*

$$G = (G_1(u), G_2(v), G_3^\varepsilon(u), G_4^\varepsilon(v)) := (B^H_u, B^H_v, B^H_{u+\varepsilon} - B^H_u, B^H_{v+\varepsilon} - B^H_v), \tag{5.11}$$

and denote

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{12} & \lambda_{22} \end{pmatrix} := \begin{pmatrix} u^{2H} & K_H(u,v) \\ K_H(v,u) & v^{2H} \end{pmatrix}^{-1} = \mathrm{Cov}^{-1}_{(G_1,G_2)}, \tag{5.12}$$

$$Q_1(u,v) := -\frac{1}{2}(\lambda_{11}G_1 + \lambda_{12}G_2), \; Q_2(u,v) := -\frac{1}{2}(\lambda_{12}G_1 + \lambda_{22}G_2). \tag{5.13}$$

a) For $\frac{1}{4} \le H < \frac{1}{3}$, as $\varepsilon \downarrow 0$,

$$\mathrm{E}\left(\frac{(G_3^\varepsilon)^3(G_4^\varepsilon)^3}{\varepsilon^2} \mid G_1, G_2\right) \overset{(law)}{=} \varepsilon^{8H-2}\left(9Q_1Q_2 - \frac{9}{4}\lambda_{12} + o(1)\right) \tag{5.14}$$

and

a') for $\frac{1}{4} \le H < \frac{1}{3}$, as $\varepsilon \downarrow 0$,

$$\mathrm{E}\left(\frac{(G_3^\varepsilon)^3}{\varepsilon} \mid G_1, G_2\right) \overset{(law)}{=} \varepsilon^{4H-1}\left(3Q_1 + o(1)\right), \mathrm{E}\left(\frac{(G_4^\varepsilon)^3}{\varepsilon} \mid G_1, G_2\right) \overset{(law)}{=} \varepsilon^{4H-1}\left(3Q_2 + o(1)\right). \tag{5.15}$$

b) Denote $G_4^\delta(v) = B_{v+\delta}^H - B_v^H$ and $G_1, G_2, G_3^\varepsilon$ as previously. Then, for $H = \frac{1}{4}$, as $\varepsilon \downarrow 0$, $\delta \downarrow 0$,

$$\mathrm{E}\left(\frac{(G_3^\varepsilon)^3(G_4^\delta)^3}{\varepsilon\delta} \mid G_1, G_2\right) \overset{(law)}{=} 9Q_1Q_2 - \frac{9}{4}\lambda_{12} + o(1). \tag{5.16}$$

c) Equivalents in (5.14), (5.15), and (5.16) are uniform on $\{1 < u, 1 < v - u\}$.

d) For $\kappa > 0$,

$$(G_1(\kappa u), G_2(\kappa v), G_3^{\kappa\varepsilon}(\kappa u), G_4^{\kappa\varepsilon}(\kappa v)) \overset{(law)}{=} \kappa^H (G_1(u), G_2(v), G_3^\varepsilon(u), G_4^\varepsilon(v)) \tag{5.17}$$

and

$$\left(G_1(\kappa u), G_2(\kappa v), Q_1(\kappa u, \kappa v)Q_2(\kappa u, \kappa v) - \frac{1}{4}\lambda_{12}(\kappa u, \kappa v)\right)$$

$$\overset{(law)}{=} \left(\kappa^H G_1(u), \kappa^H G_2(v), \kappa^{-2H}(Q_1(u,v)Q_2(u,v) - \frac{1}{4}\lambda_{12}(u,v))\right). \tag{5.18}$$

Remark 5.2 *The computation of limits when ε or (ε, δ) go to zero requires asymptotic equivalent expressions of the conditional expectations (parts a) and b) of Lemma 5.1). However, since we have to integrate on the domain $\{0 < u < v < 1\}$, we need to check that those are uniform on u, v (see part c) of Lemma 5.1).*

We postpone the proof of Lemma 5.1 and we finish the proof of (3.6). Let $0 < \rho < 1$. The second moment of $I_\varepsilon(g)$ can be written as

$$\mathrm{E}\left[\tfrac{1}{2}I_\varepsilon^2(g)\right] = \iint_{0<u<\varepsilon^{1-\rho}, u<v<1} \mathrm{E}\left\{g(G_1)g(G_2)\tfrac{(G_3^\varepsilon)^3(G_4^\varepsilon)^3}{\varepsilon^2}\right\} dudv$$

$$+ \iint_{0<v-u<\varepsilon^{1-\rho}, 0<u,v<1} \mathrm{E}\left\{g(G_1)g(G_2)\tfrac{(G_3^\varepsilon)^3(G_4^\varepsilon)^3}{\varepsilon^2}\right\} dudv$$

$$+ \iint_{\varepsilon^{1-\rho}<u<1, \varepsilon^{1-\rho}<v-u<1, v<1} \mathrm{E}\left\{g(G_1)g(G_2)\tfrac{(G_3^\varepsilon)^3(G_4^\varepsilon)^3}{\varepsilon^2}\right\} dudv$$

Using assumptions on g we can bound the first term by

$$\text{const.} \iint_{0<u<\varepsilon^{1-\rho}, u<v<1} \frac{\varepsilon^{3H}\varepsilon^{3H}}{\varepsilon^2} dudv = \text{const.}\, \varepsilon^{6H-2+1-\rho}.$$

In the sequel of this step, we will use in a significant way point *d)* of Lemma 5.1.
Choosing $0 < \rho < 6H - 1$, we can see that the first term converges to 0, as $\varepsilon \downarrow 0$. A similar reasoning implies that the second term converges also to 0. Let us denote $\varepsilon^{1-\rho} = \kappa$ and $\varepsilon^{\rho} = \tilde{\varepsilon}$ (hence $\varepsilon = \kappa\tilde{\varepsilon}$). In the third term we operate the change of variables $u = \kappa\tilde{u}$ and $v = \kappa\tilde{v}$. Hence, as $\varepsilon \downarrow 0$,

$$\iint_{\kappa<u<1,\kappa<v-u<1,v<1} \mathrm{E}\left\{g(G_1(u))g(G_2(v))\frac{(G_3^{\varepsilon}(u))^3(G_4^{\varepsilon}(v))^3}{\varepsilon^2}\right\}dudv$$

$$= \iint_{1<\tilde{u}<\frac{1}{\kappa},1<\tilde{v}-\tilde{u}<\frac{1}{\kappa},\tilde{v}<\frac{1}{\kappa}} \mathrm{E}\left\{g(G_1(\kappa\tilde{u}))g(G_2(\kappa\tilde{v}))\frac{(G_3^{\kappa\tilde{\varepsilon}}(\kappa\tilde{u}))^3(G_4^{\kappa\tilde{\varepsilon}}(\kappa\tilde{v}))^3}{\kappa^2\tilde{\varepsilon}^2}\right\}\kappa^2 d\tilde{u}d\tilde{v}$$

$$\overset{(5.17)}{=} \iint_{1<\tilde{u}<\frac{1}{\kappa},1<\tilde{v}-\tilde{u}<\frac{1}{\kappa},\tilde{v}<\frac{1}{\kappa}} \mathrm{E}\left\{g(\kappa^H G_1(\tilde{u}))g(\kappa^H G_2(\tilde{v}))\frac{\kappa^{6H}(G_3^{\tilde{\varepsilon}}(u))^3(G_4^{\tilde{\varepsilon}}(v))^3}{\tilde{\varepsilon}^2}\right\}d\tilde{u}d\tilde{v}$$

$$= \iint_{1<\tilde{u}<\frac{1}{\kappa},1<\tilde{v}-\tilde{u}<\frac{1}{\kappa},\tilde{v}<\frac{1}{\kappa}} \mathrm{E}\left\{g(\kappa^H G_1(\tilde{u}))g(\kappa^H G_2(\tilde{v}))\kappa^{6H}\right.$$
$$\left.\times\, \mathrm{E}\left(\frac{(G_3^{\tilde{\varepsilon}}(u))^3(G_4^{\tilde{\varepsilon}}(v))^3}{\tilde{\varepsilon}^2} \mid G_1(\tilde{u}), G_2(\tilde{v})\right)\right\}d\tilde{u}d\tilde{v}$$

$$\overset{(5.14)}{\sim} \iint_{1<\tilde{u}<\frac{1}{\kappa},1<\tilde{v}-\tilde{u}<\frac{1}{\kappa},\tilde{v}<\frac{1}{\kappa}} \mathrm{E}\left\{g(\kappa^H G_1(\tilde{u}))g(\kappa^H G_2(\tilde{v}))\kappa^{6H}\tilde{\varepsilon}^{8H-2}\right.$$
$$\left.\times\left(9Q_1(\tilde{u},\tilde{v})Q_2(\tilde{u},\tilde{v}) - \frac{9}{4}\lambda_{12}(\tilde{u},\tilde{v})\right)\right\}d\tilde{u}d\tilde{v}$$

$$= \iint_{\kappa<u<1,\kappa<v-u<1,v<1} \mathrm{E}\left\{g(\kappa^H G_1(\frac{u}{\kappa}))g(\kappa^H G_2(\frac{v}{\kappa}))(\kappa\tilde{\varepsilon})^{6H}\tilde{\varepsilon}^{2H-2}\right.$$
$$\left.\times\left(9Q_1(\frac{u}{\kappa},\frac{v}{\kappa})Q_2(\frac{u}{\kappa},\frac{v}{\kappa}) - \frac{9}{4}\lambda_{12}(\frac{u}{\kappa},\frac{v}{\kappa})\right)\right\}\frac{dudv}{\kappa^2}$$

$$\overset{(5.18)}{=} \iint_{\kappa<u<1,\kappa<v-u<1,v<1} \mathrm{E}\left\{g(G_1(u))g(G_2(v))(\kappa\tilde{\varepsilon})^{6H}\tilde{\varepsilon}^{2H-2}\right.$$
$$\left.\times\,\kappa^{2H-2}\left(9Q_1(u,v)Q_2(u,v) - \frac{9}{4}\lambda_{12}(u,v) + o(1)\right)\right\}dudv$$

$$= \varepsilon^{8H-2}\iint_{\kappa<u<1,\kappa<v-u<1,v<1} \mathrm{E}\left\{g(G_1(u))g(G_2(v))\right.$$
$$\left.\times\left(9Q_1(u,v)Q_2(u,v) - \frac{9}{4}\lambda_{12}(u,v) + o(1)\right)\right\}dudv,$$

where we have also used point *c)* of Lemma 5.1 to replace the conditional expectation by the uniform equivalent asymptotics in (5.14) on $\{1 < \tilde{u}, 1 < \tilde{v} - \tilde{u}\}$. Therefore, as $\varepsilon \downarrow 0$,

$$\mathrm{E}\left[I_\varepsilon(g)^2\right] \sim \varepsilon^{8H-2}\mathrm{E}\{\frac{9}{2}\iint du\,dv\,g(G_1)g(G_2)((\lambda_{11}G_1 + \lambda_{12}G_2)(\lambda_{12}G_1 + \lambda_{22}G_2) - \lambda_{12})\}.$$

¿From the expression above (3.6) can follow. Moreover

$$\lim_{\varepsilon\downarrow 0}\mathrm{E}\left[I_\varepsilon(g)^2\right] = 0, \text{ if } \frac{1}{4} < H < \frac{1}{3}, \tag{5.19}$$

which together with (5.8) gives *a)* of Theorem 3.4. $\square$

III) Absolute convergence of the integrals in (3.5) and (3.6).
The absolute convergence of the integral on the right hand side of (3.5) is already explained by the reasoning operated in *I)*. We need however to justify the absolute convergence of the integral on the right hand side of (3.6), which means

$$J := \iint_{0<u<v<1} du\, dv\, \mathrm{E}|g(B_u^{\frac{1}{4}})g(B_v^{\frac{1}{4}})$$

$$\times(\lambda_{11}\lambda_{12}(B_u^{\frac{1}{4}})^2 + (\lambda_{11}\lambda_{22}+\lambda_{12}^2)B_u^{\frac{1}{4}}B_v^{\frac{1}{4}} + \lambda_{12}\lambda_{22}(B_v^{\frac{1}{4}})^2 - \lambda_{12})| < \infty.$$

We can write $J = J_1 + J_2 + J_3 + J_4$, where

$$J_i := \iint_{0<u<v<1} \mathrm{E}(|\mathcal{E}_i(u,v)|)du\, dv,\ i = 1,2,3,\ J_4 := \iint_{0<u<v<1} \mathrm{E}(|g(B_u^{\frac{1}{4}})g(B_v^{\frac{1}{4}})\lambda_{12}|)du\, dv.$$

where

$$\begin{cases} \mathcal{E}_1(u,v) = g(B_u^{\frac{1}{4}})g(B_v^{\frac{1}{4}})\left(\lambda_{11}\lambda_{12}+\lambda_{11}\lambda_{22}+\lambda_{12}^2+\lambda_{12}\lambda_{22}\right)(B_u^{\frac{1}{4}})^2, \\ \mathcal{E}_2(u,v) = g(B_u^{\frac{1}{4}})g(B_v^{\frac{1}{4}})(\lambda_{11}\lambda_{22}+\lambda_{12}^2)B_u^{\frac{1}{4}}(B_v^{\frac{1}{4}} - B_u^{\frac{1}{4}}), \\ \mathcal{E}_3(u,v) = g(B_u^{\frac{1}{4}})g(B_v^{\frac{1}{4}})\lambda_{12}\lambda_{22}(B_v^{\frac{1}{4}} + B_u^{\frac{1}{4}})(B_v^{\frac{1}{4}} - B_u^{\frac{1}{4}}). \end{cases} \tag{5.20}$$

We set $v = u(1+\eta)$ so that

$$J_i = \textstyle\iint_{0<u<1, 0<\eta<\frac{1}{u}-1} \mathrm{E}(|\mathcal{E}_i(u,\eta)|)u\, du\, d\eta,\ i = 1,2,3,$$

$$J_4 = \textstyle\iint_{0<u<1, 0<\eta<\frac{1}{u}-1} \mathrm{E}(|g(B_u^{\frac{1}{4}})g(B_{u(1+\eta)}^{\frac{1}{4}})\lambda_{12}(u,\eta)|)u\, du\, d\eta.$$

We introduce the following notations:

$$\begin{array}{ll} K_{\frac{1}{4}}(u, u(1+\eta)) = \sqrt{u}\hat{K}(\eta), & \text{with } \hat{K}(\eta) := \frac{1}{2}(1+\sqrt{1+\eta}-\sqrt{\eta}) \\ \sqrt{u\cdot u(1+\eta)} - K_{\frac{1}{4}}^2(u,u(1+\eta)) = u\hat{\Delta}(\eta), & \text{with } \hat{\Delta}(\eta) := \sqrt{1+\eta} - \hat{K}^2(\eta). \end{array}$$

We remark that

$$\begin{array}{ll} \hat{K}(\eta) \sim 1,\ \text{ as } \eta \downarrow 0 & \text{and } \hat{K}(\eta) \sim \frac{1}{2},\ \text{ as } \eta \uparrow \infty, \\ \hat{\Delta}(\eta) \sim \sqrt{\eta},\ \text{ as } \eta \downarrow 0 & \text{or as } \eta \uparrow \infty. \end{array}$$

Using (3.7) we can write

$$\lambda_{11} = \frac{1}{\sqrt{u}}\frac{\sqrt{1+\eta}}{\hat{\Delta}(\eta)},\ \lambda_{22} = \frac{1}{\sqrt{u}}\frac{1}{\hat{\Delta}(\eta)},\ \lambda_{12} = -\frac{1}{\sqrt{u}}\frac{\hat{K}(\eta)}{\hat{\Delta}(\eta)}.$$

We can now prove that each J_i is a convergent double integral. To illustrate this fact, we prove the convergence of J_2, the computation being similar for the other integrals J_i. We recall that

$$J_2 = \textstyle\iint_{0<u<1, 0<\eta<\frac{1}{u}-1} \mathrm{E}(|\lambda_{11}\lambda_{22}+\lambda_{12}^2||g(B_u^{\frac{1}{4}})g(B_{u(1+\eta)}^{\frac{1}{4}})B_u^{\frac{1}{4}}(B_{u(1+\eta)}^{\frac{1}{4}} - B_u^{\frac{1}{4}})|)u\, du\, d\eta$$

$$= \textstyle\iint_{0<u<1, 0<\eta<\frac{1}{u}-1} \frac{\sqrt{1+\eta}+\hat{K}^2(\eta)}{\hat{\Delta}^2(\eta)} \mathrm{E}(|g(B_u^{\frac{1}{4}})g(B_{u(1+\eta)}^{\frac{1}{4}})B_u^{\frac{1}{4}}(B_{u(1+\eta)}^{\frac{1}{4}} - B_u^{\frac{1}{4}})|)du\, d\eta.$$

By Cauchy-Schwarz inequality and taking in account the assumption on g we can write

$$\mathrm{E}|g(B^{\frac{1}{4}}_u)g(B^{\frac{1}{4}}_{u(1+\eta)})B^{\frac{1}{4}}_u(B^{\frac{1}{4}}_{u(1+\eta)} - B^{\frac{1}{4}}_u)| \leq \text{const}.u^{1/2}\eta^{1/4}.$$

On the other hand

$$\frac{\sqrt{1+\eta}+\hat{K}^2(\eta)}{\hat{\Delta}^2(\eta)} \sim \frac{2}{\eta}, \text{ as } \eta \downarrow 0 \text{ and } \frac{\sqrt{1+\eta}+\hat{K}^2(\eta)}{\hat{\Delta}^2(\eta)} \sim \frac{1}{\sqrt{\eta}}, \text{ as } \eta \uparrow \infty.$$

Hence, we need now to study respectively the integrals

$$\iint_{0<u<1,0<\eta<1} \tfrac{u^{1/2}}{\eta^{3/4}}\, du\, d\eta < \infty,$$

$$\iint_{0<u<1,1<\eta<\frac{1}{u}-1} \tfrac{u^{1/2}}{\eta^{1/4}}\, du\, d\eta = \int_1^\infty \tfrac{d\eta}{\eta^{1/4}} \int_0^{1/(\eta+1)} u^{1/2} du = \tfrac{2}{3}\int_1^\infty \tfrac{d\eta}{\eta^{1/4}(\eta+1)^{3/2}} < \infty.$$

This concludes the proof of point *c)* of Theorem 3.4. □

IV) Proof of the forward third order integral existence.
IV-1) Reduction to the case of a bounded function g
Suppose for a moment that we know the result when g is bounded. Since the paths of $B^{\frac{1}{4}}$ are continuous, we prove by localization that the result is true when g is only locally bounded. Let $\alpha > 0$. We will show that $\{I_\varepsilon(g) : \varepsilon > 0\}$ is Cauchy with respect to the convergence in probability, i.e.

$$\lim_{\varepsilon\downarrow 0,\delta\downarrow 0} \mathrm{P}(|I_\varepsilon(g) - I_\delta(g)| \geq \alpha) = 0.$$

Let $M > 0$, $\Omega_M = \{|B^{\frac{1}{4}}_u| \leq M; \forall u \in [0, t+1]\}$. On Ω_M, we have $I_\varepsilon(g) = I_\varepsilon(g_M)$ and $I_\delta(g) = I_\delta(g_M)$ where g_M is a function with compact support, which coincides on g on the compact interval $[-M, M]$.

Therefore, $\mathrm{P}(\{|I_\varepsilon(g) - I_\delta(g)| \geq \alpha\} \cap \Omega^c_M) \leq \mathrm{P}(\Omega^c_M)$. We choose M large enough, so that $P(\Omega^c_M)$ is uniformly small with respect to ε and δ. Then

$$\begin{aligned}\mathrm{P}(\{|I_\varepsilon(g) - I_\delta(g)| \geq \alpha\} \cap \Omega_M) &= \mathrm{P}(\{|I_\varepsilon(g_M) - I_\delta(g_M)| \geq \alpha\} \cap \Omega_M)\\ &\leq \mathrm{P}(|I_\varepsilon(g_M) - I_\delta(g_M)| \geq \alpha).\end{aligned}$$

Since g_M has compact support, $I_\varepsilon(g_M)$ converges in probability.

IV-2) Proof of the existence when g is a bounded function
Thus, it remains to prove that the sequence $\{I_\varepsilon(g) : \varepsilon > 0\}$ converges in probability, when g is bounded. For this purpose, we even show that, in that case, the sequence is even Cauchy in $\mathrm{L}^2(\Omega)$.

We will prove the Cauchy criterium for $\{I_\varepsilon(g) : \varepsilon > 0\}$:

$$\lim_{\varepsilon\downarrow 0,\delta\downarrow 0} \mathrm{E}\left(|I_\varepsilon(g) - I_\delta(g)|^2\right) = \lim_{\varepsilon\downarrow 0,\delta\downarrow 0} \mathrm{E}\left[I_\varepsilon(g)^2\right] + \mathrm{E}\left[I_\delta(g)^2\right] - 2\mathrm{E}\left[I_\varepsilon(g)I_\delta(g)\right] = 0.$$

The first two terms converge to the same limit given in (3.6) as $\varepsilon \downarrow 0$ and $\delta \downarrow 0$. It remains to show that $\lim_{\varepsilon\downarrow 0,\delta\downarrow 0} \mathrm{E}\left[I_\varepsilon(g)I_\delta(g)\right]$ equals to the right hand-side of (3.6), and then the Cauchy criterium will be fulfilled. A simple change of variable gives,

$$I_\varepsilon(g)I_\delta(g) = \iint_{0<u<v<1} g(B_u^{\frac14})g(B_v^{\frac14})\,\frac{(B_{u+\varepsilon}^{\frac14}-B_u^{\frac14})^3}{\varepsilon}\frac{(B_{v+\delta}^{\frac14}-B_v^{\frac14})^3}{\delta}\,dudv$$

$$+\iint_{0<u<v<1} g(B_u^{\frac14})g(B_v^{\frac14})\,\frac{(B_{u+\delta}^{\frac14}-B_u^{\frac14})^3}{\delta}\frac{(B_{v+\varepsilon}^{\frac14}-B_v^{\frac14})^3}{\varepsilon}\,dudv.$$

Taking the expectation of the expression above gives

$$\lim_{\varepsilon\downarrow 0,\delta\downarrow 0}\mathrm{E}\left[I_\varepsilon(g)I_\delta(g)\right] = 2\lim_{\varepsilon\downarrow 0,\delta\downarrow 0}\mathrm{E}\left\{\iint_{0<u<v<1} g(G_1)g(G_2)\mathrm{E}\left(\frac{(G_3^\varepsilon)^3(G_4^\delta)}{\varepsilon\delta}\mid G_1,G_2\right)dudv\right\}$$

so that the result will be a consequence of (5.16). □

V) Proof of the existence of a Hölder continuous version.

It is enough to show the existence of a continuous version for $t\in[0,T]$, for any $T>0$. Suppose for a moment that for every g bounded we can show the existence of a (Hölder) continuous version for $(\int_0^t g(B_u^{\frac14})d^{-3}B_u^{\frac14})_{t\in[0,T]}$. We denote it by $(\tilde I(g)_t)_{t\in[0,T]}$. Then, we can define the associated version for a general $g\in L^\infty_{\text{loc}}(\mathbb{R})$, by

$$\tilde I(g)(\omega) = \tilde I(g^M)(\omega),$$

where $g^M = g\mathbf{1}_{[-M,M]}$, if $\omega\in\{\sup_{t\in[0,T]} : |B_t^{\frac14}|\le M\}$.

Therefore, it remains to prove that the forward third order integral has a Hölder continuous version (with Hölder parameter less than $\frac14$), when g is bounded and continuous.

We prove that the L^2-valued function $t\mapsto I(g)(t) := \int_0^t g(B_u^{\frac14})d^{-3}B_u^{\frac14}$ has a Hölder continuous version on $[0,T]$. We need to control, for $s<t$, s,t in compact intervals,

$$\mathrm{E}\left[(I(g)(t)-I(g)(s))^2\right] = \mathrm{E}\left[\left(\int_s^t g(B_u^{\frac14})d^{-3}B_u^{\frac14}\right)^2\right]$$

$$\le \iint_{s\le u<v\le t} du\,dv\,\mathrm{E}[|g(B_u^{\frac14})g(B_v^{\frac14})||\mathcal{E}_1(u,v)+\mathcal{E}_2(u,v)+\mathcal{E}_3(u,v)-\lambda_{12}|],$$

where $\mathcal{E}_i(u,v)$, $i=1,2,3$, are given by (5.20). Let us denote

$$\mathcal{E}_1(u,v) = \tilde{\mathcal{E}}_1(u,v)(B_u^{\frac14})^2,\ \mathcal{E}_2(u,v) = \tilde{\mathcal{E}}_2(u,v)B_u^{\frac14}(B_v^{\frac14}-B_u^{\frac14}),\ \mathcal{E}_3(u,v) = \tilde{\mathcal{E}}_3(u,v)(B_v^{\frac14}+B_u^{\frac14})(B_v^{\frac14}-B_u^{\frac14}).$$

We denote again $\eta = v-u$. Therefore

$$\tilde{\mathcal{E}}_1(u,u+\eta) = \lambda_{11}\lambda_{12}+\lambda_{11}\lambda_{22}+\lambda_{12}^2+\lambda_{12}\lambda_{22} = \frac{1}{2\Delta^2}\eta\frac{\sqrt{u}}{\sqrt{u+\eta}+\sqrt{u}} = \frac{1}{2\Delta^2}\eta\frac{\sqrt{u/\eta}}{\sqrt{1+u/\eta}+\sqrt{u/\eta}},$$

$$\tilde{\mathcal{E}}_2(u,u+\eta) = \lambda_{11}\lambda_{22}+\lambda_{12}^2 = \frac{1}{2\Delta^2}\left(u+\eta+3\sqrt{u}\sqrt{u+\eta}-\sqrt{u}\sqrt{\eta}-\sqrt{\eta}\sqrt{u+\eta}\right),$$

$$\tilde{\mathcal{E}}_3(u,u+\eta) = \lambda_{12}\lambda_{22} = -\frac{1}{2\Delta^2}u\left(1+\frac{\sqrt{u}}{\sqrt{u+\eta}+\sqrt{\eta}}\right) = -\frac{1}{2\Delta^2}u\left(1+\frac{\sqrt{u/\eta}}{1+\sqrt{1+u/\eta}}\right)$$

$$-\lambda_{12} = \frac{1}{2\Delta}\sqrt{u}\left(1+\frac{\sqrt{u}}{\sqrt{u+\eta}+\sqrt{\eta}}\right) = \frac{1}{2\Delta}\sqrt{u}\left(1+\frac{\sqrt{u/\eta}}{1+\sqrt{1+u/\eta}}\right),$$

where

$$\Delta := \sqrt{u(u+\eta)} - K_H^2(u,u+\eta) = \frac{1}{2}\sqrt{u}\sqrt{\eta}\left(1 + \frac{\sqrt{u}}{\sqrt{u+\eta}+\sqrt{\eta}} + \frac{\sqrt{\eta}}{\sqrt{u+\eta}+\sqrt{u}}\right) \geq \frac{1}{2}\sqrt{u}\sqrt{\eta}.$$

The functions $\psi_1(x) = \frac{\sqrt{x}}{\sqrt{x}+\sqrt{1+x}}$ and respectively $\psi_2(x) = \frac{\sqrt{x}}{1+\sqrt{1+x}}$ are positive increasing on $[0,+\infty[$ with limit $\frac{1}{2}$, respectively 1 as $x \uparrow \infty$. Moreover, we see that $\sqrt{u+\eta} \leq \sqrt{u} + \sqrt{\eta}$. Therefore

$$0 \leq \tilde{\mathcal{E}}_1(u,u+\eta) \leq \frac{1}{u},\ |\tilde{\mathcal{E}}_2(u,u+\eta)| \leq \frac{8}{\eta} + \frac{4}{u} + \frac{10}{\sqrt{u}\sqrt{\eta}},\ |\tilde{\mathcal{E}}_2(u,u+\eta)| \leq \frac{4}{\eta},\ 0 \leq -\lambda_{12} \leq \frac{2}{\sqrt{\eta}}.$$

Hence

$$\iint_{s\leq u<v\leq t} \mathrm{E}\left[|g(B_u^{\frac{1}{4}})g(B_v^{\frac{1}{4}})||\tilde{\mathcal{E}}_1(u,v)|(B_u^{\frac{1}{4}})^2\right] du\,dv \leq \text{const.} \iint_{s\leq u\leq t, 0<\eta\leq t-s} \frac{du d\eta}{\sqrt{u}} = \text{const.}(t-s)^{\frac{3}{2}},$$

$$\iint_{s\leq u<v\leq t} \mathrm{E}\left[|g(B_u^{\frac{1}{4}})g(B_v^{\frac{1}{4}})||\tilde{\mathcal{E}}_2(u,v)|\,|B_u^{\frac{1}{4}}(B_v^{\frac{1}{4}} - B_u^{\frac{1}{4}})|\right] du\,dv$$

$$\leq \text{const.} \iint_{s\leq u\leq t, 0<\eta\leq t-s} \left(8\frac{u^{\frac{1}{4}}}{\eta^{\frac{3}{4}}} + 4\frac{\eta^{\frac{1}{4}}}{u^{\frac{3}{4}}} + 10\frac{1}{u^{\frac{1}{4}}\eta^{\frac{1}{4}}}\right) du d\eta$$

$$= \text{const.}(8(t^{\frac{5}{4}} - s^{\frac{5}{4}})(t-s)^{\frac{1}{4}} + 4(t^{\frac{1}{4}} - s^{\frac{1}{4}})(t-s)^{\frac{5}{4}} + 10(t^{\frac{3}{4}} - s^{\frac{3}{4}})(t-s)^{\frac{3}{4}}) \leq \text{const.}(t-s)^{\frac{3}{2}-\rho},$$

where $\rho > 0$,

$$\iint_{s\leq u<v\leq t} \mathrm{E}\left[|g(B_u^{\frac{1}{4}})g(B_v^{\frac{1}{4}})||\mathcal{E}_3(u,v)|\,|(B_v^{\frac{1}{4}} + B_u^{\frac{1}{4}})(B_v^{\frac{1}{4}} - B_u^{\frac{1}{4}})|\right] du\,dv$$

$$\leq \text{const.} \iint_{s\leq u\leq t, 0<\eta\leq t-s} \frac{du d\eta}{\sqrt{u}\,\eta^{\frac{3}{4}}} = \text{const.}(t-s)^{\frac{5}{4}}$$

and

$$\iint_{s\leq u<v\leq t} \mathrm{E}\left[|g(B_u^{\frac{1}{4}})g(B_v^{\frac{1}{4}})|\,|\lambda_{12}|\right] du\,dv \leq \text{const.}(t-s)^{\frac{3}{2}}.$$

Therefore

$$\mathrm{E}\left[(I(g)(t) - I(g)(s))^2\right] \leq \text{const.}(t-s)^{1+\frac{1}{2}-\rho}, \ \text{with}\ \rho > 0$$

The classical Kolmogorov criterion allows then to conclude. □

VI) Proof of (3.4) and point d).

It is not easy to make computations or to recognize the positivity using the right-hand side of the second moment of the third order integrals, see (3.6). We need to give other expression of the second moment but also to compute their covariance with the integral in point *d)*. This will be possible when g is smooth. Using Proposition 3.6 and an obvious approximation argument it is enough to suppose that $g \in C^1(\mathbb{R})$ with g and g' bounded.

Since the third order integrals are continuous to prove (3.4) we need only to verify that for fixed $t > 0$

$$\mathrm{E}\left(\int_0^t g(B_u^{\frac{1}{4}})d^{\pm 3}B_u^{\frac{1}{4}} \mp \frac{3}{2}\int_0^t g'(B_u^{\frac{1}{4}})du\right)^2 = 0. \tag{5.21}$$

This equality is a simple consequence of the following lemma:

Lemma 5.3 *Let g, h be real functions, $g \in C^1(\mathbb{R})$ and h locally bounded such that g, g', h fulfill the subexponential inequality (3.3). The following equalities holds:*

$$\mathrm{E}\left\{\left(\int_0^t g(B_u^{\frac{1}{4}})d^{\pm 3}B_u^{\frac{1}{4}}\right)^2\right\} = \frac{9}{4}\mathrm{E}\left\{\left(\int_0^t g'(B_u^{\frac{1}{4}})du\right)^2\right\} \tag{5.22}$$

and

$$\mathrm{E}\left\{\left(\int_0^t g(B_u^{\frac{1}{4}})d^{\pm 3}B_u^{\frac{1}{4}}\right)\left(\int_0^t h(B_u^{\frac{1}{4}})du\right)\right\} = \mp\frac{3}{2}\mathrm{E}\left\{\left(\int_0^t g'(B_u^{\frac{1}{4}})du\right)\left(\int_0^t h(B_u^{\frac{1}{4}})du\right)\right\}. \tag{5.23}$$

Finally, by (2.15) we also get the statement in the point *d)*. □

This achieves the proof of Theorem 3.4 and we can proceed to the proof of Lemma 5.3.

Proof of (5.22) in Lemma 5.3. To simplify notations, we write K for $K_{\frac{1}{4}}(u,v)$ and Δ for $\sqrt{uv} - K^2$. Hence

$$\lambda_{11} = \frac{\sqrt{v}}{\Delta},\ \lambda_{22} = \frac{\sqrt{u}}{\Delta},\ \lambda_{12} = -\frac{K}{\Delta}.$$

Let us introduce the matrix

$$M = \begin{pmatrix} u^{\frac{1}{4}} & 0 \\ \frac{K}{u^{\frac{1}{4}}} & \frac{\sqrt{\Delta}}{u^{\frac{1}{4}}} \end{pmatrix}, \text{ with } M^{-1} = \begin{pmatrix} u^{\frac{1}{4}} & 0 \\ -u^{-\frac{1}{4}}\frac{K}{\sqrt{\Delta}} & \frac{u^{\frac{1}{4}}}{\sqrt{\Delta}} \end{pmatrix}$$

and observe that, by (5.12), MM^* is the covariance matrix of $(B_u^{\frac{1}{4}}, B_v^{\frac{1}{4}})$. Furthermore, if N_1, N_2 are two independent standard normal random variables, then $\begin{pmatrix} B_u^{\frac{1}{4}} \\ B_v^{\frac{1}{4}} \end{pmatrix} = M\begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$. After some algebraic computations, we obtain

$$\lambda_{11}\lambda_{12}(B_u^{\frac{1}{4}})^2 + (\lambda_{11}\lambda_{22} + \lambda_{12}^2)B_u^{\frac{1}{4}}B_v^{\frac{1}{4}} + \lambda_{12}\lambda_{22}(B_v^{\frac{1}{4}})^2 - \lambda_{12}$$

$$= \left((M^{-1})^*\begin{pmatrix} N_1 \\ N_2 \end{pmatrix}\right)_1 \cdot \left((M^{-1})^*\begin{pmatrix} N_1 \\ N_2 \end{pmatrix}\right)_2 - \left((M^{-1})^*M^{-1}\right)_{12} = \tfrac{N_1N_2}{\sqrt{\Delta}} - \tfrac{K N_2^2}{\Delta} + \tfrac{K}{\Delta}.$$

Therefore, by (3.6), for $t = 1$,

$$\mathrm{E}\left\{\left(\int_0^1 g(B_u^{\frac{1}{4}})d^{-3}B_u^{\frac{1}{4}}\right)^2\right\}$$

$$= \tfrac{9}{2}\iint_{0<u<v<1} du\, dv \mathrm{E}\left[g(u^{\frac{1}{4}}N_1)g\left(\tfrac{K}{u^{\frac{1}{4}}}N_1 + \tfrac{\sqrt{\Delta}}{u^{\frac{1}{4}}}N_2\right)\left(\tfrac{N_1N_2}{\sqrt{\Delta}} - \tfrac{K N_2^2}{\Delta} + \tfrac{K}{\Delta}\right)\right]$$

$$= \tfrac{9}{2}\iint_{0<u<v<1} du\, dv \mathrm{E}\left[g'(u^{\frac{1}{4}}N_1)g'\left(\tfrac{K}{u^{\frac{1}{4}}}N_1 + \tfrac{\sqrt{\Delta}}{u^{\frac{1}{4}}}N_2\right)\right] = \tfrac{9}{4}\,\mathrm{E}\left\{\left(\int_0^1 g'(B_u^{\frac{1}{4}})du\right)^2\right\}.$$

The second equality is given by the following identity, for $a, b, c \in \mathbb{R}$, $a > 0$,

$$\mathrm{E}\left[g(aN_1)g\left(\frac{b}{a}N_1+\frac{c}{a}N_2\right)\left(\frac{1}{c}N_1N_2-\frac{b}{c^2}\left(N_2^2-1\right)\right)\right]=\mathrm{E}\left[g'(aN_1)g'\left(\frac{b}{a}N_1+\frac{c}{a}N_2\right)\right], \tag{5.24}$$

which can be obtained by direct calculation, using Gaussian densities, the assumption on g and integration by parts.

This concludes the proof of (5.22). □

Proof of (5.23) in Lemma 5.3. We verify nowa more general covariance type equality between the third order integral $\int_0^1 g(B_u^{\frac{1}{4}})d^{-3}B_u^{\frac{1}{4}}$ with a random variable of the form $\int_0^t h(B_u^{\frac{1}{4}})du$:

Let g, h be real locally bounded functions fulfilling the subexponential inequality (3.3). Then

$$\mathrm{E}\left\{(\int_0^t g(B_u^{\frac{1}{4}})d^{-3}B_u^{\frac{1}{4}})(\int_0^t h(B_u^{\frac{1}{4}})du)\right\}=-\frac{3}{2}\mathrm{E}\left\{\int_0^t dv\int_0^t du\, g(B_u^{\frac{1}{4}})h(B_v^{\frac{1}{4}})(\lambda_{11}B_u^{\frac{1}{4}}+\lambda_{12}B_v^{\frac{1}{4}})\right\} \tag{5.25}$$

Before verifying this result, wes prove (5.23). Taking again $t=1$, (5.25) implies that the left member of (5.23) equals

$$-\frac{3}{2}\int_0^1 dv\int_0^1 du\, g(B_u^{\frac{1}{4}})h(B_v^{\frac{1}{4}})\left(\frac{\sqrt{v}}{\Delta}B_u^{\frac{1}{4}}-\frac{K}{\Delta}B_v^{\frac{1}{4}}\right), \tag{5.26}$$

where we denote again $K=K_{\frac{1}{4}}(u,v)$, $\Delta=\sqrt{uv}-K^2$. As in the proof of (5.22), we can write

$$B_u^{\frac{1}{4}}=u^{\frac{1}{4}}N_1,\ B_v^{\frac{1}{4}}=\frac{K}{u^{\frac{1}{4}}}N_1+\frac{\sqrt{\Delta}}{u^{\frac{1}{4}}}N_2,$$

where N_1, N_2 are again independent $N(0,1)$ random variables. Therefore (5.27) gives

$$-\frac{3}{2}\int_0^1 dv\int_0^1 du\,\mathrm{E}\left\{g(u^{\frac{1}{4}}N_1)\,h\left(\frac{K}{u^{\frac{1}{4}}}N_1+\frac{\sqrt{\Delta}}{u^{\frac{1}{4}}}N_2\right)\left[\frac{N_1}{u^{\frac{1}{4}}}-\frac{K}{\sqrt{\Delta}u^{\frac{1}{4}}}\right]\right\}. \tag{5.27}$$

Similarly to identity (5.24), we can establish the following, for $a, b, c \in \mathbb{R}$, $a > 0$:

$$\mathrm{E}\left(g(aN_1)\,h\left(\frac{b}{a}N_1+\frac{c}{a}N_2\right)\left(\frac{N_1}{a}-\frac{b}{ac}N_2\right)\right)=\mathrm{E}\left(g'(aN_1)\,h\left(\frac{b}{a}N_1+\frac{c}{a}N_2\right)\right). \tag{5.28}$$

The proof follows easily again using integration by parts. We apply (5.28) with $a=u^{\frac{1}{4}}$, $b=K$, $c=\sqrt{\Delta}$. Hence, (5.27) gives

$$-\frac{3}{2}\int_0^1 dv\int_0^1 du\,\mathrm{E}\left\{g'(u^{\frac{1}{4}}N_1)\,h\left(\frac{K}{u^{\frac{1}{4}}}N_1+\frac{\sqrt{\Delta}}{u^{\frac{1}{4}}}N_2\right)\right)=-\frac{3}{2}\int_0^1 dv\int_0^1 du\,\mathrm{E}\left\{g'(B_u^{\frac{1}{4}})\,h(B_v^{\frac{1}{4}})\right\},$$

that is the right member of (5.23).

We come back to the proof of (5.25) and we follow a similar reasoning as for the evaluation of the second moment of the third order integral, see point *c)* of Theorem 3.4. Since $\int_0^1 g(B_u^{\frac{1}{4}})d^{-3}B_u^{\frac{1}{4}}$ is the limit in $\mathrm{L}^2(\Omega)$ of $I_\varepsilon(g)$, then

$$\mathrm{E}\left(\int_0^1 g(B_u^{\frac{1}{4}})d^{-3}B_u^{\frac{1}{4}}\int_0^1 h(B_v^{\frac{1}{4}})dv\right) \text{ is the limit of } J_\varepsilon^1 + J_\varepsilon^2,$$

where

$$J_\varepsilon^1 := \frac{1}{\varepsilon}\int_0^1 dv\int_0^v du\mathrm{E}\left(g(B_u^{\frac{1}{4}})(B_{u+\varepsilon}^{\frac{1}{4}} - B_u^{\frac{1}{4}})^3 h(B_v^{\frac{1}{4}})\right) = \int_0^1\int_0^v du\mathrm{E}\left(g(G_1)h(G_2)\frac{(G_3^\varepsilon)^3}{\varepsilon}\right),$$

$$J_\varepsilon^2 := \frac{1}{\varepsilon}\int_0^1 dv\int_0^v du\mathrm{E}\left(g(B_v^{\frac{1}{4}})(B_{v+\varepsilon}^{\frac{1}{4}} - B_v^{\frac{1}{4}})^3 h(B_u^{\frac{1}{4}})\right) = \int_0^1\int_0^v du\mathrm{E}\left(g(G_2)h(G_1)\frac{(G_4^\varepsilon)^3}{\varepsilon}\right)$$

using the same notations as for the evaluation of the second moment at point *c)*.

We can write

$$J_\varepsilon^1 = \int_0^1\int_0^v du\mathrm{E}\left\{g(G_1)h(G_2)\mathrm{E}\left(\frac{(G_3^\varepsilon)^3}{\varepsilon} \mid G_1, G_2\right)\right\}$$

$$= -\frac{3}{2}\left\{\mathrm{E}\left[g(G_1)h(G_2)\left(\lambda_{11}G_1 - \lambda_{12}G_2\right)\right] + o(1)\right\},$$

by the point *a')* of Lemma 5.1, since $H = \frac{1}{4}$. Moreover, by the point *c)* of the same lemma the estimates are uniform in u and v. Therefore Lebesgue dominated convergence theorem says that

$$\lim_{\varepsilon\downarrow 0} J_\varepsilon^1 = -\frac{3}{2}\int_0^1 dv\int_0^v du\,\mathrm{E}\left[g(B_u^{\frac{1}{4}})h(B_v^{\frac{1}{4}})(\lambda_{11}B_u^{\frac{1}{4}} + \lambda_{12}B_v^{\frac{1}{4}})\right].$$

Proceeding similarly for J_ε^2, using again Lemma 5.1, we obtain

$$\lim_{\varepsilon\downarrow 0} J_\varepsilon^2 = -\frac{3}{2}\int_0^1 dv\int_0^v du\,\mathrm{E}\left[g(B_v^{\frac{1}{4}})h(B_u^{\frac{1}{4}})(\lambda_{12}B_u^{\frac{1}{4}} + \lambda_{22}B_v^{\frac{1}{4}})\right]$$

$$= -\frac{3}{2}\int_0^1 dv\int_v^1 du\,\mathrm{E}\left[g(B_u^{\frac{1}{4}})h(B_v^{\frac{1}{4}})(\lambda_{12}B_v^{\frac{1}{4}} + \lambda_{11}B_u^{\frac{1}{4}})\right].$$

Finally

$$\lim_{\varepsilon\downarrow 0}(J_\varepsilon^1 + J_\varepsilon^2) = -\frac{3}{2}\int_0^1 dv\int_0^1 du\,\mathrm{E}\left[g(B_u^{\frac{1}{4}})h(B_v^{\frac{1}{4}})(\lambda_{11}B_u^{\frac{1}{4}} + \lambda_{12}B_v^{\frac{1}{4}})\right],$$

which is the desired quantity. □

This achieves the proof of Lemma 5.3 and we can proceed to the proof of Lemma 5.1:

Proof of point *a)* in Lemma 5.1. We write the covariance matrix of $(G_1, G_2, G_3^\varepsilon, G_4^\varepsilon)$ by blocks:

$$\Lambda_\varepsilon = \begin{pmatrix}\Lambda_{11} & \Lambda_{12}^\varepsilon \\ \Lambda_{21}^\varepsilon & \Lambda_{22}^\varepsilon\end{pmatrix}.$$

By classical Gaussian analysis, we know that the matrix A_ε and the covariance matrix of the vector Z^ε in (IV.1) can be expressed as:

$$A_\varepsilon = \Lambda_{21}^\varepsilon\Lambda_{11}^{-1} \text{ and } K_{Z^\varepsilon} = \Lambda_{22}^\varepsilon - A_\varepsilon(\Lambda_{21}^\varepsilon)^*. \tag{5.29}$$

Here,

$$\Lambda_{11} = \begin{pmatrix} u^{2H} & K_H(u,v) \\ K_H(v,u) & v^{2H} \end{pmatrix}, \Lambda_{21}^{\varepsilon} = \begin{pmatrix} \alpha_\varepsilon(u)u^{2H} & \gamma_\varepsilon(u,v) \\ \gamma_\varepsilon(v,u) & \alpha_\varepsilon(v)v^{2H} \end{pmatrix}, \Lambda_{22}^{\varepsilon} = \begin{pmatrix} \varepsilon^{2H} & \eta_\varepsilon(u,v) \\ \eta_\varepsilon(v,u) & \varepsilon^{2H} \end{pmatrix}, \tag{5.30}$$

where α_ε is given by (5.4) and

$$\gamma_\varepsilon(u,v) := \mathrm{Cov}(G_3^\varepsilon, G_2) = \tfrac{1}{2}\left((u+\varepsilon)^{2H} - u^{2H} - |v-u-\varepsilon|^{2H} + |v-u|^{2H}\right),$$

$$\eta_\varepsilon(u,v) := \mathrm{Cov}(G_3^\varepsilon, G_4^\varepsilon) = \tfrac{1}{2}\left(|v-u+\varepsilon|^{2H} + |v-u-\varepsilon|^{2H} - 2|v-u|^{2H}\right).$$

Also recall that $\Lambda_{11}^{-1} = (\lambda_{ij})_{i,j=1,2}$ is the inverse of the covariance matrix of (G_1, G_2) (see (5.12)). We can see that

$$\gamma_\varepsilon(u,v) = H\left(u^{2H-1} + |v-u|^{2H-1}\right)\varepsilon + o(\varepsilon) \ \text{ as } \ \varepsilon \downarrow 0, \tag{5.31}$$

and

$$\eta_\varepsilon(u,v) = H(2H-1)|v-u|^{2H-2}\varepsilon^2 + o(\varepsilon^2), \ \text{ as } \ \varepsilon \downarrow 0. \tag{5.32}$$

We split the proof in several steps.

Step 1: expansion of the matrix A_ε.
We express its components by

$$A_\varepsilon := \begin{pmatrix} a_{11}^\varepsilon & a_{12}^\varepsilon \\ a_{21}^\varepsilon & a_{22}^\varepsilon \end{pmatrix}. \tag{5.33}$$

Using (5.6), (5.29) and (5.31), when $\varepsilon \downarrow 0$, gives

$$a_{11}^\varepsilon = \lambda_{11}\alpha_\varepsilon(u)u^{2H} + \lambda_{12}\gamma_\varepsilon(u,v) = -\frac{\lambda_{11}}{2}\varepsilon^{2H} + H\left((\lambda_{11}+\lambda_{12})u^{2H-1} + \lambda_{12}|v-u|^{2H-1}\right)\varepsilon + o(\varepsilon). \tag{5.34}$$

The asymptotics of the other coefficients a_{ij}^ε behaves similarly, since

$$a_{12}^\varepsilon = \lambda_{12}\alpha_\varepsilon(u)u^{2H} + \lambda_{22}\gamma_\varepsilon(u,v),\ a_{21}^\varepsilon = \lambda_{12}\alpha_\varepsilon(v)v^{2H} + \lambda_{11}\gamma_\varepsilon(v,u),$$

$$a_{22}^\varepsilon = \lambda_{22}\alpha_\varepsilon(v)v^{2H} + \lambda_{12}\gamma_\varepsilon(v,u).$$

The expansion as $\varepsilon \downarrow 0$ for the matrix A_ε becomes

$$A_\varepsilon = \begin{pmatrix} -\frac{\lambda_{11}}{2}\varepsilon^{2H} + k_{11}\varepsilon + o(\varepsilon) & -\frac{\lambda_{12}}{2}\varepsilon^{2H} + k_{12}\varepsilon + o(\varepsilon) \\ \\ -\frac{\lambda_{12}}{2}\varepsilon^{2H} + k_{21}\varepsilon + o(\varepsilon) & -\frac{\lambda_{22}}{2}\varepsilon^{2H} + k_{22}\varepsilon + o(\varepsilon) \end{pmatrix}, \tag{5.35}$$

where $k_{ij} := k_{ij}(u,v)\ i,j = 1,2$,

$$\begin{pmatrix} k_{11}(u,v) & k_{12}(u,v) \\ k_{21}(u,v) & k_{22}(u,v) \end{pmatrix} = \tag{5.36}$$

$$H\begin{pmatrix}(\lambda_{11}+\lambda_{12})u^{2H-1}+\lambda_{12}|v-u|^{2H-1} & (\lambda_{12}+\lambda_{22})u^{2H-1}+\lambda_{22}|v-u|^{2H-1}\\ \\ (\lambda_{12}+\lambda_{11})v^{2H-1}+\lambda_{11}|u-v|^{2H-1} & (\lambda_{22}+\lambda_{12})v^{2H-1}+\lambda_{12}|u-v|^{2H-1}\end{pmatrix}.$$

Step 2: expansion of the matrix K_{Z^ε}.
We claim that the expansion of the matrix K_{Z^ε} when $\varepsilon\downarrow 0$, is:

$$K_{Z^\varepsilon}=\begin{pmatrix}K_{Z^\varepsilon}(1,1) & K_{Z^\varepsilon}(1,2)\\ \\ K_{Z^\varepsilon}(1,2) & K_{Z^\varepsilon}(2,2)\end{pmatrix}, \tag{5.37}$$

with

$$\begin{cases} K_{Z^\varepsilon}(1,1)=\varepsilon^{2H}-\frac{\lambda_{11}}{4}\varepsilon^{4H}+k_{11}\varepsilon^{1+2H}+o(\varepsilon^{1+2H})\\ \\ K_{Z^\varepsilon}(1,2)=-\frac{\lambda_{12}}{4}\varepsilon^{4H}+\frac{k_{12}+k_{21}}{2}\varepsilon^{1+2H}+o(\varepsilon^{1+2H})\\ \\ K_{Z^\varepsilon}(2,2)=\varepsilon^{2H}-\frac{\lambda_{22}}{4}\varepsilon^{4H}+k_{22}\varepsilon^{1+2H}+o(\varepsilon^{1+2H}).\end{cases} \tag{5.38}$$

We compute K_{Z^ε} explicitely. Clearly, the computations for $K_{Z^\varepsilon}(1,1)$ and $K_{Z^\varepsilon}(2,2)$ are similar. Using (5.29)-(5.32) and (5.35), for $\varepsilon\downarrow 0$,

$$\begin{aligned}
&K_{Z^\varepsilon}(1,1)=\varepsilon^{2H}-a^\varepsilon_{11}\alpha_\varepsilon(u)u^{2H}-a^\varepsilon_{12}\gamma_\varepsilon(u,v)\\
&=\varepsilon^{2H}-\varepsilon^{4H}\left(-\tfrac{\lambda_{11}}{2}+k_{11}\varepsilon^{1-2H}+o(\varepsilon^{1-2H})\right)\left(-\tfrac12+Hu^{2H-1}\varepsilon^{1-2H}+o(\varepsilon^{1-2H})\right)\\
&-\varepsilon^{1+2H}\left(-\tfrac{\lambda_{12}}{2}+k_{12}\varepsilon^{1-2H}+o(\varepsilon^{1-2H})\right)\left(H\left(u^{2H-1}+|v-u|^{2H-1}\right)+o(1)\right)\\
&=\varepsilon^{2H}-\varepsilon^{4H}\left(\tfrac{\lambda_{11}}{4}-(\tfrac{\lambda_{11}}{2}Hu^{2H-1}+\tfrac{k_{11}}{2})\varepsilon^{1-2H}+o(\varepsilon^{1-2H})\right)\\
&-\varepsilon^{1+2H}\left(-\tfrac{\lambda_{12}}{2}H\left(u^{2H-1}+|v-u|^{2H-1}\right)+o(1)\right)=\varepsilon^{2H}-\tfrac{\lambda_{11}}{4}\varepsilon^{4H}+k_{11}\varepsilon^{1+2H}+o(\varepsilon^{1+2H}),
\end{aligned}$$

whereas

$$\begin{aligned}
&K_{Z^\varepsilon}(1,2)=\eta_\varepsilon(u,v)-a^\varepsilon_{12}\alpha_\varepsilon(v)v^{2H}-a^\varepsilon_{11}\gamma_\varepsilon(v,u)=\varepsilon^2\left(H(2H-1)|v-u|^{2H-2}+o(1)\right)\\
&-\varepsilon^{4H}\left(-\tfrac{\lambda_{12}}{2}+k_{12}\varepsilon^{1-2H}+o(\varepsilon^{1-2H})\right)\left(-\tfrac12+Hv^{2H-1}\varepsilon^{1-2H}+o(\varepsilon^{1-2H})\right)\\
&-\varepsilon^{1+2H}\left(-\tfrac{\lambda_{11}}{2}+k_{11}\varepsilon^{1-2H}+o(\varepsilon^{1-2H})\right)\left(H\left(v^{2H-1}+|v-u|^{2H-1}\right)+o(1)\right)\\
&=-\tfrac{\lambda_{12}}{4}\varepsilon^{4H}+\tfrac{k_{12}+k_{21}}{2}\varepsilon^{1+2H}+o(\varepsilon^{1+2H}).
\end{aligned}$$

Step 3: the law of the vector Z^ε.
Using (5.37) and (5.38) we observe that the Gaussian vector Z^ε can be written as

$$\begin{pmatrix}Z_1^\varepsilon\\ Z_2^\varepsilon\end{pmatrix}\overset{\text{(law)}}{=}\begin{pmatrix}\nu(\varepsilon)N_1\\ \mu(\varepsilon)N_1+\theta(\varepsilon)N_2\end{pmatrix}, \tag{5.39}$$

where N_1, N_2 are independent standard normal random variables, independent also of G_1, G_2. Moreover, for $\varepsilon \downarrow 0$,

$$\begin{cases} \nu(\varepsilon) = \varepsilon^H \left(1 - \frac{\lambda_{11}}{8}\varepsilon^{2H} + c_1\varepsilon + o(\varepsilon)\right),\ \mu(\varepsilon) = \varepsilon^{3H}\left(-\frac{\lambda_{12}}{4} + c_2\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right) \\ \theta(\varepsilon) = \varepsilon^H \left(1 - \frac{\lambda_{22}}{8}\varepsilon^{2H} + c_3\varepsilon + o(\varepsilon)\right), \end{cases} \tag{5.40}$$

where $c_i := c_i(u,v)$, $i = 1,2,3$,

$$c_1(u,v) := \begin{cases} \frac{k_{11}}{2}, & \text{if } H > \frac{1}{4} \\ \frac{k_{11}}{2} - \frac{\lambda_{11}^2}{128}, & \text{if } H = \frac{1}{4} \end{cases},\quad c_2(u,v) := \begin{cases} \frac{k_{12}+k_{21}}{2}, & \text{if } H > \frac{1}{4} \\ \frac{k_{12}+k_{21}}{2} - \frac{\lambda_{11}\lambda_{12}}{32}, & \text{if } H = \frac{1}{4} \end{cases},$$

and

$$c_3(u,v) := \begin{cases} \frac{k_{22}}{2}, & \text{if } H > \frac{1}{4} \\ \frac{k_{22}}{2} + \frac{\lambda_{12}^2}{32} - \frac{\lambda_{22}^2}{128}, & \text{if } H = \frac{1}{4} \end{cases}.$$

Indeed, when $\varepsilon \downarrow 0$,

$$\nu(\varepsilon) = \sqrt{K_{Z^\varepsilon}(1,1)} = \varepsilon^H\left(1 - \frac{\lambda_{11}}{4}\varepsilon^{2H} + k_{11}\varepsilon + o(\varepsilon)\right)^{1/2} = \varepsilon^H\Big(1 - \frac{\lambda_{11}}{8}\varepsilon^{2H} + \frac{k_{11}}{2}\varepsilon$$

$$-\frac{\lambda_{11}^2}{128}\varepsilon^{4H} + o(\varepsilon)\Big) = \begin{cases} \varepsilon^H\left(1 - \frac{\lambda_{11}}{8}\varepsilon^{2H} + \frac{k_{11}}{2}\varepsilon + o(\varepsilon)\right), & \text{if } H > \frac{1}{4} \\ \varepsilon^H\left(1 - \frac{\lambda_{11}}{8}\varepsilon^{2H} + (\frac{k_{11}}{2} - \frac{\lambda_{11}^2}{128})\varepsilon + o(\varepsilon)\right), & \text{if } H = \frac{1}{4} \end{cases},$$

$$\mu(\varepsilon) = \frac{K_{Z^\varepsilon}(1,2)}{\nu(\varepsilon)} = \frac{\varepsilon^{4H}\left(-\frac{\lambda_{12}}{4} + \frac{k_{12}+k_{21}}{2}\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right)}{\varepsilon^H\left(1 - \frac{\lambda_{11}}{8}\varepsilon^{2H} + c_1\varepsilon + o(\varepsilon)\right)}$$

$$= \varepsilon^{3H}\left(-\frac{\lambda_{12}}{4} + \frac{k_{12}+k_{21}}{2}\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right)\left(1 + \frac{\lambda_{11}}{8}\varepsilon^{2H} - c_1\varepsilon + \frac{\lambda_{11}^2}{64}\varepsilon^{4H} + o(\varepsilon)\right)$$

$$= \varepsilon^{3H}\left(-\frac{\lambda_{12}}{4} - \frac{\lambda_{11}\lambda_{12}}{32}\varepsilon^{2H} + \frac{k_{12}+k_{21}}{2}\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right)$$

$$= \begin{cases} \varepsilon^{3H}\left(-\frac{\lambda_{12}}{4} + \frac{k_{12}+k_{21}}{2}\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right), & \text{if } H > \frac{1}{4} \\ \varepsilon^{3H}\left(-\frac{\lambda_{12}}{4} + (\frac{k_{12}+k_{21}}{2} - \frac{\lambda_{11}\lambda_{12}}{32})\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right), & \text{if } H = \frac{1}{4} \end{cases},$$

and

$$\theta(\varepsilon) = \sqrt{K_{Z^\varepsilon}(2,2) - \mu^2(\varepsilon)}$$

$$= \sqrt{\varepsilon^{2H} - \frac{\lambda_{22}}{4}\varepsilon^{4H} + k_{22}\varepsilon^{1+2H} + o(\varepsilon^{1+2H}) - \varepsilon^{6H}\left(-\frac{\lambda_{12}}{4} + c_2\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right)^2}$$

$$= \sqrt{\varepsilon^{2H} - \frac{\lambda_{22}}{4}\varepsilon^{4H} + k_{22}\varepsilon^{1+2H} - \frac{\lambda_{12}^2}{16}\varepsilon^{6H} + o(\varepsilon^{1+2H})}$$

$$= \varepsilon^H \left(1 - \frac{\lambda_{22}}{4}\varepsilon^{2H} + k_{22}\varepsilon - \frac{\lambda_{12}^2}{16}\varepsilon^{4H} + o(\varepsilon)\right)^{1/2} = \varepsilon^H \left(1 - \frac{\lambda_{22}}{8}\varepsilon^{2H} + \frac{k_{22}}{2}\varepsilon\right.$$

$$\left. -(\frac{\lambda_{12}^2}{32} + \frac{\lambda_{22}^2}{128})\varepsilon^{4H} + o(\varepsilon)\right) = \begin{cases} \varepsilon^H \left(1 - \frac{\lambda_{22}}{8}\varepsilon^{2H} + \frac{k_{22}}{2}\varepsilon + o(\varepsilon)\right), & \text{if } H > \frac{1}{4} \\ \varepsilon^H \left(1 - \frac{\lambda_{22}}{8}\varepsilon^{2H} + (\frac{k_{22}}{2} - \frac{\lambda_{12}^2}{32} - \frac{\lambda_{22}^2}{128})\varepsilon + o(\varepsilon)\right), & \text{if } H = \frac{1}{4}. \end{cases}$$

Step 4: the law of the vector $(G_3^\varepsilon, G_4^\varepsilon)$.
We claim that, for $\varepsilon \downarrow 0$,

$$\begin{pmatrix} G_3^\varepsilon \\ G_4^\varepsilon \end{pmatrix} \overset{\text{(law)}}{=} \begin{pmatrix} N_1\varepsilon^H + Q_1\varepsilon^{2H} - \frac{\lambda_{11}}{8}N_1\varepsilon^{3H} + R_1\varepsilon + o(\varepsilon) \\ N_2\varepsilon^H + Q_2\varepsilon^{2H} - (\frac{\lambda_{12}}{4}N_1 + \frac{\lambda_{22}}{8}N_2)\varepsilon^{3H} + R_2\varepsilon + o(\varepsilon) \end{pmatrix}, \tag{5.41}$$

where

$$R_1 := k_{11}G_1 + k_{12}G_2 \ \text{ and } \ R_2 := k_{21}G_1 + k_{22}G_2.$$

Indeed, using (5.33), (5.35), (5.39) and (5.40), when $\varepsilon \downarrow 0$, we get

$$\begin{cases} G_3^\varepsilon = a_{11}^\varepsilon G_1 + a_{12}^\varepsilon G_2 + Z_1^\varepsilon \overset{\text{(law)}}{=} (-\frac{\lambda_{11}}{2}\varepsilon^{2H} + k_{11}\varepsilon + o(\varepsilon))G_1 + (-\frac{\lambda_{12}}{2}\varepsilon^{2H} + k_{12}\varepsilon + o(\varepsilon))G_2 \\ +\varepsilon^H(1 - \frac{\lambda_{11}}{8}\varepsilon^{2H} + c_1\varepsilon + o(\varepsilon))N_1, \\ G_4^\varepsilon = a_{21}^\varepsilon G_1 + a_{22}^\varepsilon G_2 + Z_2^\varepsilon \overset{\text{(law)}}{=} (-\frac{\lambda_{12}}{2}\varepsilon^{2H} + k_{21}\varepsilon + o(\varepsilon))G_1 + (-\frac{\lambda_{22}}{2}\varepsilon^{2H} + k_{22}\varepsilon + o(\varepsilon))G_2 \\ +\varepsilon^{3H}(-\frac{\lambda_{12}}{4} + c_2\varepsilon^{1-2H} + o(\varepsilon^{1-2H}))N_1 + \varepsilon^H(1 - \frac{\lambda_{22}}{8}\varepsilon^{2H} + c_3\varepsilon + o(\varepsilon))N_2. \end{cases}$$

Step 5: evaluation of the law of $G_3^\varepsilon G_4^\varepsilon$.
As a consequence of previous step,

$$G_3^\varepsilon G_4^\varepsilon \overset{\text{(law)}}{=} \varepsilon^{2H}(N_1 + Q_1\varepsilon^H - \frac{\lambda_{11}}{8}N_1\varepsilon^{2H} + R_1\varepsilon^{1-H} + o(\varepsilon^{1-H}))(N_2 + Q_2\varepsilon^H$$

$$-(\frac{\lambda_{12}}{4}N_1 + \frac{\lambda_{22}}{8}N_2)\varepsilon^{2H} + R_2\varepsilon^{1-H} + o(\varepsilon^{1-H})) \overset{\text{(law)}}{=} \varepsilon^{2H}(N_1N_2 + (N_1Q_2 + N_2Q_1)\varepsilon^H$$

$$+(Q_1Q_2 - \frac{\lambda_{12}}{4}N_1^2 - \frac{\lambda_{11}+\lambda_{22}}{8}N_1N_2)\varepsilon^{2H} + o(\varepsilon^{2H})) \overset{\text{(law)}}{=} \varepsilon^{2H}\left(N_1N_2 + S_\varepsilon\right),$$

where

$$S_\varepsilon \overset{\text{(law)}}{=} \varepsilon^H(N_1Q_2 + N_2Q_1 + (Q_1Q_2 - \frac{\lambda_{12}}{4}N_1^2 - \frac{\lambda_{11} + \lambda_{22}}{8}N_1N_2)\varepsilon^H + o(\varepsilon^H)).$$

Step 6: evaluation of the law of $(G_3^\varepsilon G_4^\varepsilon)^3$.
We observe that, when $\varepsilon \downarrow 0$,

$$S_\varepsilon^2 \overset{\text{(law)}}{=} \varepsilon^{2H}((N_1Q_2 + N_2Q_1)^2 + o(1)) \ \text{ and } \ S_\varepsilon^3 \overset{\text{(law)}}{=} o(\varepsilon^{3H}).$$

Hence

$$(G_3^\varepsilon G_4^\varepsilon)^3 \overset{\text{(law)}}{=} \left\{\varepsilon^{2H}(N_1N_2+S_\varepsilon)\right\}^3 \overset{\text{(law)}}{=} \varepsilon^{6H}\left(N_1^3N_2^3+3N_1^2N_2^2S_\varepsilon+3N_1N_2S_\varepsilon^2+S_\varepsilon^3\right)$$

$$\overset{\text{(law)}}{=} \varepsilon^{6H}\{N_1^3N_2^3+3N_1^2N_2^2[N_1Q_2+N_2Q_1]\varepsilon^H$$

$$+[9N_1^2N_2^2Q_1Q_2-\tfrac{3\lambda_{12}}{4}N_1^4N_2^2-3\tfrac{\lambda_{11}+\lambda_{22}}{8}N_1^3N_2^3+3N_1^3N_2Q_2^2+3N_1N_2^3Q_1^2]\varepsilon^{2H}+o(\varepsilon^{2H})\}.$$

Step 7: computation of the conditional expectation in (5.14).
Consequently, for $\varepsilon\downarrow 0$

$$\tfrac{(G_3^\varepsilon)^3(G_4^\varepsilon)^3}{\varepsilon^2} \overset{\text{(law)}}{=} \varepsilon^{6H-2}\{N_1^3N_2^3+[3N_1^3N_2^2Q_2+3N_1^2N_2^3Q_1]\varepsilon^H$$

$$+[9N_1^2N_2^2Q_1Q_2-\tfrac{3\lambda_{12}}{4}N_1^4N_2^2-3\tfrac{\lambda_{11}+\lambda_{22}}{8}N_1^3N_2^3+3N_1^3N_2Q_2^2+3N_1N_2^3Q_1^2]\varepsilon^{2H}+o(\varepsilon^{2H})\}.$$

Since N_1, N_2 are independent standard normal random variables, also independent of G_1, G_2, we obtain the conditional expectation in (5.14). □

Proof of *b)* **of Lemma 5.1.** The proof is similar as for *a)*. We will only provide the most significant arguments in several steps. Asymptotics for $\varepsilon\downarrow 0$, $\delta\downarrow 0$ of some functions of (ε,δ) in fractional powers can be done using a Maple procedure (see **http://www.iecn.u-nancy.fr/~gradinar/procalc.ps**). Equalities involving such a procedure are indicated by $(\star)$. Recall that the Hurst index is here $H=\frac{1}{4}$.

Step 1: linear regression.
We can write

$$\begin{pmatrix} G_3^\varepsilon \\ G_4^\delta \end{pmatrix} = A_{\varepsilon,\delta}\begin{pmatrix} G_1 \\ G_2 \end{pmatrix} + \begin{pmatrix} Z_1^{\varepsilon,\delta} \\ Z_2^{\varepsilon,\delta} \end{pmatrix}, \tag{5.42}$$

with

$$A_{\varepsilon,\delta}=\Lambda_{12}^{\varepsilon,\delta}\Lambda_{11}^{-1} \text{ and } K_{Z^{\varepsilon,\delta}}=\Lambda_{22}^{\varepsilon,\delta}-A_{\varepsilon,\delta}(\Lambda_{12}^{\varepsilon,\delta})^*. \tag{5.43}$$

Here

$$\Lambda_{12}^{\varepsilon,\delta}=\begin{pmatrix} \alpha_\varepsilon(u)\sqrt{u} & \gamma_\varepsilon(u,v) \\ \gamma_\delta(v,u) & \alpha_\delta(v)\sqrt{v} \end{pmatrix}, \Lambda_{22}^{\varepsilon,\delta}=\begin{pmatrix} \varepsilon^{\frac{1}{2}} & \eta_{\varepsilon,\delta}(u,v) \\ \eta_{\varepsilon,\delta}(u,v) & \delta^{\frac{1}{2}} \end{pmatrix}, \tag{5.44}$$

with

$$\eta_{\varepsilon,\delta}(u,v)=\mathrm{Cov}(G_3^\varepsilon,G_4^\delta)=\frac{1}{2}\left(|v-u+\delta|^{\frac{1}{2}}+|v-u-\varepsilon|^{\frac{1}{2}}-|v-u|^{\frac{1}{2}}-|v-u+\delta-\varepsilon|^{\frac{1}{2}}\right).$$

Therefore, when $\varepsilon\downarrow 0, \delta\downarrow 0$

$$\eta_{\varepsilon,\delta}(u,v)=-\frac{\varepsilon\delta}{8|v-u|^{\frac{3}{2}}}+o((\varepsilon+\delta)^2). \tag{5.45}$$

Step 2: expansion and computations for the matrix $A_{\varepsilon,\delta}$.
We can write

$$A_{\varepsilon,\delta} := \begin{pmatrix} a^{\varepsilon}_{11} & a^{\varepsilon}_{12} \\ a^{\delta}_{21} & a^{\delta}_{22} \end{pmatrix},$$

with

$$\begin{array}{ll} a^{\varepsilon}_{11} = \lambda_{11}\alpha_{\varepsilon}(u)\sqrt{u} + \lambda_{12}\gamma_{\varepsilon}(u,v), & a^{\varepsilon}_{12} = \lambda_{12}\alpha_{\varepsilon}(u)\sqrt{u} + \lambda_{22}\gamma_{\varepsilon}(u,v), \\ a^{\delta}_{21} = \lambda_{12}\alpha_{\delta}(v)\sqrt{v} + \lambda_{11}\gamma_{\delta}(v,u), & a^{\delta}_{22} = \lambda_{22}\alpha_{\delta}(v)\sqrt{v} + \lambda_{12}\gamma_{\delta}(v,u). \end{array}$$

Hence, as in part *a) step 1)*, as $\varepsilon \downarrow 0$, $\delta \downarrow 0$,

$$A_{\varepsilon,\delta} = \begin{pmatrix} -\frac{\lambda_{11}}{2}\varepsilon^{\frac{1}{2}} + k_{11}\varepsilon + o(\varepsilon) & -\frac{\lambda_{12}}{2}\varepsilon^{\frac{1}{2}} + k_{12}\varepsilon + o(\varepsilon) \\ \\ -\frac{\lambda_{12}}{2}\delta^{\frac{1}{2}} + k_{21}\delta + o(\delta) & -\frac{\lambda_{22}}{2}\delta^{\frac{1}{2}} + k_{22}\delta + o(\delta) \end{pmatrix} \tag{5.46}$$

where, k_{ij} are given by (5.36).

Step 3: computations related to matrix $K_{Z^{\varepsilon,\delta}}$.
We can write

$$K_{Z^{\varepsilon,\delta}} = \begin{pmatrix} K_{Z^{\varepsilon,\delta}}(1,1) & K_{Z^{\varepsilon,\delta}}(1,2) \\ \\ K_{Z^{\varepsilon,\delta}}(1,2) & K_{Z^{\varepsilon,\delta}}(2,2) \end{pmatrix}, \tag{5.47}$$

where if $\varepsilon \downarrow 0$, $\delta \downarrow 0$, we have

$$\begin{cases} K_{Z^{\varepsilon,\delta}}(1,1) = \varepsilon^{\frac{1}{2}} - \frac{\lambda_{11}}{4}\varepsilon + k_{11}\varepsilon^{\frac{3}{2}} + o(\varepsilon^{\frac{3}{2}}),\ K_{Z^{\varepsilon,\delta}}(2,2) = \delta^{\frac{1}{2}} - \frac{\lambda_{22}}{4}\delta + k_{22}\delta^{\frac{3}{2}} + o(\delta^{\frac{3}{2}}) \\ \\ K_{Z^{\varepsilon,\delta}}(1,2) = -\frac{\lambda_{12}}{4}\varepsilon^{\frac{1}{2}}\delta^{\frac{1}{2}} + \frac{k_{12}}{2}\varepsilon\delta^{\frac{1}{2}} + \frac{k_{21}}{2}\varepsilon^{\frac{1}{2}}\delta + o((\varepsilon+\delta)^2). \end{cases} \tag{5.48}$$

Indeed,

$$K_{Z^{\varepsilon,\delta}}(1,1) = \varepsilon^{\frac{1}{2}} - a^{\varepsilon}_{11}\alpha_{\varepsilon}(u)\sqrt{u} - a^{\varepsilon}_{12}\gamma_{\varepsilon}(u,v) \ \text{ and } \ K_{Z^{\varepsilon,\delta}}(2,2) = \delta^{\frac{1}{2}} - a^{\delta}_{22}\alpha_{\delta}(v)\sqrt{v} - a^{\delta}_{21}\gamma_{\delta}(v,u),$$

hence the expansions of those two coefficients are similar as in part *a) step 3)*. The expansion of the remaining element behaves differently. Indeed, for $\varepsilon \downarrow 0$, $\delta \downarrow 0$,

$$K_{Z^{\varepsilon,\delta}}(1,2) = \eta_{\varepsilon,\delta}(u,v) - a^{\varepsilon}_{12}\alpha_{\delta}(v)\sqrt{v} - a^{\varepsilon}_{11}\gamma_{\delta}(v,u) = -\frac{\varepsilon\delta}{8|v-u|^{\frac{3}{2}}} + o((\varepsilon+\delta)^2)$$

$$-\varepsilon^{\frac{1}{2}}\delta^{\frac{1}{2}}\left(-\frac{\lambda_{12}}{2} + k_{12}\varepsilon^{\frac{1}{2}} + o(\varepsilon^{\frac{1}{2}})\right)\left(-\frac{1}{2} + \frac{1}{4\sqrt{v}}\delta^{\frac{1}{2}} + o(\delta^{\frac{1}{2}})\right) - \varepsilon^{\frac{1}{2}}\delta\left(-\frac{\lambda_{11}}{2} + k_{11}\varepsilon^{\frac{1}{2}} + o(\varepsilon^{\frac{1}{2}})\right)$$

$$\times\left(\frac{1}{4\sqrt{v}} + \frac{1}{4\sqrt{|u-v|}} + o(1)\right) \overset{(\star)}{=} \dots = -\frac{\lambda_{12}}{4}\varepsilon^{\frac{1}{2}}\delta^{\frac{1}{2}} + \frac{k_{12}}{2}\varepsilon\delta^{\frac{1}{2}} + \frac{k_{21}}{2}\varepsilon^{\frac{1}{2}}\delta + o((\varepsilon+\delta)^2).$$

Step 4: the law of the vector $(Z^{\varepsilon,\delta}_1, Z^{\varepsilon,\delta}_2)$.
Computations give

$$\begin{pmatrix} Z^{\varepsilon,\delta}_1 \\ Z^{\varepsilon,\delta}_2 \end{pmatrix} \overset{\text{(law)}}{=} \begin{pmatrix} \nu(\varepsilon)N_1 \\ \mu(\varepsilon,\delta)N_1 + \theta(\varepsilon,\delta)N_2 \end{pmatrix}, \tag{5.49}$$

where N_1, N_2 are independent standard normal random variables, also independent of G_1, G_2 and where

$$\begin{cases} \nu(\varepsilon) = \varepsilon^{\frac{1}{4}} - \frac{\lambda_{11}}{8}\varepsilon^{\frac{3}{4}} + o(\varepsilon^{\frac{3}{4}}),\ \mu(\varepsilon,\delta) = -\frac{\lambda_{12}}{4}\varepsilon^{\frac{1}{4}}\delta^{\frac{1}{2}} + o((\varepsilon^{\frac{1}{2}} + \delta^{\frac{1}{2}})^2), \\ \theta(\varepsilon,\delta) = \delta^{\frac{1}{4}} + \frac{\lambda_{12}}{8}\varepsilon^{\frac{1}{4}}\delta^{\frac{1}{4}} - \frac{\lambda_{22}}{8}\delta^{\frac{3}{4}} - \frac{\lambda_{12}^2}{128}\varepsilon^{\frac{1}{2}}\delta^{\frac{1}{4}} + \left(\frac{\lambda_{11}\lambda_{12}}{64} - \frac{k_{12}}{4} + \frac{\lambda_{12}^3}{1024}\right)\varepsilon^{\frac{3}{4}}\delta^{\frac{1}{4}} \\ + \left(\frac{\lambda_{12}\lambda_{22}}{64} - \frac{k_{21}}{4}\right)\varepsilon^{\frac{1}{4}}\delta^{\frac{3}{4}} + o(\varepsilon^{\frac{1}{2}} + \delta^{\frac{1}{2}}). \end{cases} \tag{5.50}$$

Indeed, $\nu(\varepsilon)$ is given by the first equality in (5.40), when $\varepsilon \downarrow 0$. The other coefficients are given by

$$\mu(\varepsilon,\delta) = \frac{K_{Z^{\varepsilon,\delta}}(1,2)}{\nu(\varepsilon)},\ \theta(\varepsilon,\delta) = \sqrt{K_{Z^{\varepsilon,\delta}}(2,2) - \mu^2(\varepsilon,\delta)}$$

and we use the results in the previous step. Hence, we have

$$\begin{cases} Z_1^{\varepsilon,\delta} \overset{(\text{law})}{=} N_1\varepsilon^{\frac{1}{4}} - \frac{\lambda_{11}}{8}N_1\varepsilon^{\frac{3}{4}} + o(\varepsilon^{\frac{3}{4}}) \\ Z_2^{\varepsilon,\delta} \overset{(\text{law})}{=} N_2\delta^{\frac{1}{4}} + \frac{\lambda_{12}}{8}N_2\varepsilon^{\frac{1}{4}}\delta^{\frac{1}{4}} - \frac{\lambda_{12}^2}{128}N_2\varepsilon^{\frac{1}{2}}\delta^{\frac{1}{4}} - \frac{\lambda_{12}}{4}N_1\varepsilon^{\frac{1}{4}}\delta^{\frac{1}{2}} - \frac{\lambda_{22}}{8}N_2\delta^{\frac{3}{4}} \\ + \left(\frac{\lambda_{11}\lambda_{12}}{64} - \frac{k_{12}}{4} + \frac{\lambda_{12}^3}{1024}\right)N_2\varepsilon^{\frac{3}{4}}\delta^{\frac{1}{4}} + \left(\frac{\lambda_{12}\lambda_{22}}{64} - \frac{k_{21}}{4}\right)N_2\varepsilon^{\frac{1}{4}}\delta^{\frac{3}{4}} + o((\varepsilon^{\frac{1}{2}} + \delta^{\frac{1}{2}})^2), \end{cases}$$

Step 5: the law of the vector $(G_3^\varepsilon, G_4^\delta)$.
Using the first line of (5.41), (5.50) and (5.48), when $\varepsilon \downarrow 0$, $\delta \downarrow 0$, we obtain

$$\begin{cases} G_3^\varepsilon \overset{(\text{law})}{=} N_1\varepsilon^{\frac{1}{4}} + Q_1\varepsilon^{\frac{1}{2}} - \frac{\lambda_{11}}{8}N_1\varepsilon^{\frac{3}{4}} + o(\varepsilon^{\frac{3}{4}}), \\ G_4^\delta \overset{(\text{law})}{=} \overset{(\star)}{\cdots} \overset{(\text{law})}{=} N_2\delta^{\frac{1}{4}} + Q_2\delta^{\frac{1}{2}} - \frac{\lambda_{22}}{8}N_2\delta^{\frac{3}{4}} + R_2\delta + \frac{\lambda_{12}}{8}N_2\varepsilon^{\frac{1}{4}}\delta^{\frac{1}{4}} - \frac{\lambda_{12}^2}{128}N_2\varepsilon^{\frac{1}{2}}\delta^{\frac{1}{4}} - \frac{\lambda_{12}}{4}N_1\varepsilon^{\frac{1}{4}}\delta^{\frac{1}{2}} \\ + \left(\frac{\lambda_{11}\lambda_{12}}{64} - \frac{k_{12}}{4} + \frac{\lambda_{12}^3}{1024}\right)N_2\varepsilon^{\frac{3}{4}}\delta^{\frac{1}{4}} + \left(\frac{\lambda_{12}\lambda_{22}}{64} - \frac{k_{21}}{4}\right)N_2\varepsilon^{\frac{1}{4}}\delta^{\frac{3}{4}} + o((\varepsilon^{\frac{1}{2}} + \delta^{\frac{1}{2}})^2), \end{cases} \tag{5.51}$$

with Q_1, Q_2 given by (5.13) and R_2 is as in part *a) step 4)*.

Step 6: computation of the law of $G_3^\varepsilon G_4^\delta$.
¿From (5.51), when $\varepsilon \downarrow 0$, $\delta \downarrow 0$, we get

$$G_3^\varepsilon G_4^\delta \overset{(\text{law})}{=} \overset{(\star)}{\cdots} \overset{(\text{law})}{=} N_1N_2\varepsilon^{\frac{1}{4}}\delta^{\frac{1}{4}} + \left(\frac{\lambda_{12}}{8}N_1N_2 + Q_1N_2\right)\varepsilon^{\frac{1}{2}}\delta^{\frac{1}{4}} + N_1Q_2\varepsilon^{\frac{1}{4}}\delta^{\frac{1}{2}} + \Big(-\frac{\lambda_{12}^2}{128}N_1N_2$$

$$-\frac{\lambda_{11}}{8}N_1N_2 + \frac{\lambda_{12}}{8}Q_1N_2\Big)\varepsilon^{\frac{3}{4}}\delta^{\frac{1}{4}} + \left(-\frac{\lambda_{12}}{4}N_1^2 + Q_1Q_2\right)\varepsilon^{\frac{1}{2}}\delta^{\frac{1}{2}} - \frac{\lambda_{22}}{8}N_1N_2\varepsilon^{\frac{1}{4}}\delta^{\frac{3}{4}} + o((\varepsilon^{\frac{1}{2}} + \delta^{\frac{1}{2}})^2).$$

Step 7: Computation of the conditional expectation in (5.16).

When $\varepsilon \downarrow 0$, $\delta \downarrow 0$, it follows

$$\frac{(G_3^\varepsilon)^3(G_4^\delta)^3}{\varepsilon\delta} \overset{\text{(law)}}{=} \overset{(\star)}{\cdots} \overset{\text{(law)}}{=} N_1^3N_2^3\varepsilon^{-\frac14}\delta^{-\frac14} + 3\left(\tfrac{\lambda_{12}}{8}N_1^3N_2^3 + Q_1N_1^2N_2^3\right)\delta^{-\frac14} + 3Q_2N_1^3N_2^2\varepsilon^{-\frac14}$$

$$+3\left(\left(\tfrac{\lambda_{12}^2}{128} - \tfrac{\lambda_{11}}{8}\right)N_1^3N_2^3 + Q_1^2N_1N_2^3 + \tfrac{3\lambda_{12}}{8}Q_1N_1^2N_2^3\right)\varepsilon^{\frac14}\delta^{-\frac14}$$

$$+3\left(-\tfrac{\lambda_{22}}{8}N_1^3N_2^3 + Q_2^2N_1^3N_2\right)\varepsilon^{-\frac14}\delta^{\frac14} + \tfrac{3\lambda_{12}}{4}Q_2N_1^3N_2^2 - \tfrac{3\lambda_{12}}{4}N_1^4N_2^2 + 9Q_1Q_2N_1^2N_2^2 + o(1)$$

Since N_1, N_2 are independent standard normal random variables, independent of G_1, G_2, we deduce finally the conditional expectation in (5.16). $\square$

Proof of *a')* **in Lemma 5.1.** Using notations (5.9), we recall that

$$G_3^\varepsilon = \left[A_\varepsilon\begin{pmatrix}G_1\\G_2\end{pmatrix}\right]_1 + Z_1^\varepsilon,\ G_4^\varepsilon = \left[A_\varepsilon\begin{pmatrix}G_1\\G_2\end{pmatrix}\right]_2 + Z_2^\varepsilon.$$

Therefore, the left member of the first equality in (5.15) can be written as

$$\mathrm{E}\left(\frac{(G_3^\varepsilon)^3}{\varepsilon} \mid G_1, G_2\right) = \mathrm{E}\left(\left[A_\varepsilon\begin{pmatrix}G_1\\G_2\end{pmatrix}\right]_1 + Z_1^\varepsilon\right)^3 = \left[A_\varepsilon\begin{pmatrix}G_1\\G_2\end{pmatrix}\right]_1^3 + 3\left[A_\varepsilon\begin{pmatrix}G_1\\G_2\end{pmatrix}\right]_1 \mathrm{E}\left[(Z_1^\varepsilon)^2\right]$$
$$= (a_{11}^\varepsilon G_1 + a_{12}^\varepsilon G_2)^3 + 3(a_{11}^\varepsilon G_1 + a_{12}^\varepsilon G_2)K_{Z^\varepsilon}(1,1),$$

according to the notations in (5.33), (5.34) and (5.37). Recall that, by (5.35) and (5.36),

$$\begin{cases} a_{11}^\varepsilon = -\frac{\lambda_{11}}{2}\varepsilon^{2H} + k_{11}\varepsilon + o(\varepsilon), & \text{where } k_{11} = H(\lambda_{11}+\lambda_{12})u^{2H-1} - \lambda_{12}H|u-v|^{2H-1} \\ a_{12}^\varepsilon = -\frac{\lambda_{12}}{2}\varepsilon^{2H} + k_{12}\varepsilon + o(\varepsilon), & \text{where } k_{12} = H(\lambda_{12}+\lambda_{22})u^{2H-1} - \lambda_{22}H|u-v|^{2H-1}, \end{cases}$$

and by (5.38), $K_{Z^\varepsilon}(1,1) = \varepsilon^{2H} - \frac{\lambda_{11}}{4}\varepsilon^{4H} + o(\varepsilon)$. Hence, we obtain

$$\mathrm{E}\left(\tfrac{(G_3^\varepsilon)^3}{\varepsilon} \mid G_1, G_2\right) = \left\{\left[-\tfrac{\lambda_{11}}{2}\varepsilon^{2H} + k_{11}\varepsilon + o(\varepsilon)\right]G_1 + \left[-\tfrac{\lambda_{12}}{2}\varepsilon^{2H} + k_{12}\varepsilon + o(\varepsilon)\right]G_2\right\}^3$$

$$+3\left\{\left[-\tfrac{\lambda_{11}}{2}\varepsilon^{2H} + k_{11}\varepsilon + o(\varepsilon)\right]G_1 + \left[-\tfrac{\lambda_{12}}{2}\varepsilon^{2H} + k_{12}\varepsilon + o(\varepsilon)\right]G_2\right\}(\varepsilon^{2H} - \tfrac{\lambda_{11}}{4}\varepsilon^{4H} + o(\varepsilon))$$

$$= 3\varepsilon^{4H}\left[-\lambda_{11}\tfrac{G_1}{2} - \lambda_{12}\tfrac{G_2}{2}\right] + o(\varepsilon).$$

This gives (5.15). $\square$

Proof of *c)* **in Lemma 5.1.** We need to show that the asymptotics in (5.14), (5.15) and (5.16) are uniform in u and v. We do the job for (5.14), the others behaving similarly. It is enough to analyze the uniformity of the expansions on $\{1 < u, 1 < v-u\}$ of $\alpha_\varepsilon(u)$, $\gamma_\varepsilon(u,v)$ and $\eta_\varepsilon(u,v)$, when $\varepsilon \downarrow 0$, because the other asymptotics are obtained in terms of those ones. When $\varepsilon \downarrow 0$, by (5.4) we have

$$\alpha_\varepsilon(u) = \frac{1}{2u^{2H}}\left((u+\varepsilon)^{2H} - u^{2H} - \varepsilon^{2H}\right) = \frac12\left(\left(1+\frac{\varepsilon}{u}\right)^{2H} - 1 - \left(\frac{\varepsilon}{u}\right)^{2H}\right) = -\frac12\left(\frac{\varepsilon}{u}\right)^{2H} + H\frac{\varepsilon}{u} + o\left(\frac{\varepsilon}{u}\right);$$

this provides a uniform expansion on $\{u > 1\}$. Similarly, when $\varepsilon \downarrow 0$, one obtains

$$\gamma_\varepsilon(u,v) = \tfrac{1}{2}\left((u+\varepsilon)^{2H} - u^{2H} - |v-u-\varepsilon|^{2H} + (v-u)^{2H}\right) = \tfrac{1}{2}\left[u^{2H}\left(\left(1+\tfrac{\varepsilon}{u}\right)^{2H} - 1\right)\right.$$

$$\left. -(v-u)^{2H}\left(\left|1-\tfrac{\varepsilon}{v-u}\right|^{2H} - 1\right)\right] = H\left(u^{2H-1} + |v-u|^{2H-1}\right)\varepsilon + o(\varepsilon)$$

uniformly on $\{1 < u, 1 < v-u\}$ and when $\varepsilon \downarrow 0$,

$$\eta_\varepsilon(u,v) = \tfrac{1}{2}\left((v-u+\varepsilon)^{2H} + |v-u-\varepsilon|^{2H} - 2(v-u)^{2H}\right)$$

$$= \tfrac{(v-u)^{2H}}{2}\left[\left(1+\tfrac{\varepsilon}{v-u}\right)^{2H} + \left|1-\tfrac{\varepsilon}{v-u}\right|^{2H} - 2\right] = H(2H-1)|v-u|^{2H-2}\varepsilon^2 + o(\varepsilon^2),$$

uniformly on $\{1 < v-u\}$. □

Proof of *d)* **in Lemma 5.1.** We look for the homogeneity degree of all quantities used so far. For a function $f = f(\varepsilon, u, v)$ we shall denote

$$\deg_{\varepsilon,u,v}(f) =: p \Leftrightarrow f(\kappa\varepsilon, \kappa u, \kappa v) = \kappa^p f(\varepsilon, u, v).$$

where we make the convention that

$$\gamma(\varepsilon,u,v) := \gamma_\varepsilon(u,v),\ K_Z(i,j)(\varepsilon,u,v) := K_{Z^\varepsilon}(i,j)(u,v).$$

We have:

$$\begin{array}{l}
\deg_{\varepsilon,u}(\alpha) = 0, \text{ by (5.4)},\\
\deg_{\varepsilon,u,v}(\lambda_{ij}) = -2H, \text{ by (5.12)},\\
\deg_{\varepsilon,u,v}(\gamma) = 2H, \text{ by (5.31)},\\
\deg_{\varepsilon,u,v}(\eta) = 2H, \text{ by (5.32)},\\
\deg_{\varepsilon,u,v}(a_{ij}) = 0, \text{ by (5.29), (5.30) and (5.33)},\\
\deg_{\varepsilon,u,v}(K_Z(i,j)) = 2H, \text{ by (5.29), (5.30) and (5.37)},\\
\deg_{\varepsilon,u,v}(\nu) = \deg_{\varepsilon,u,v}(\mu) = \deg_{\varepsilon,u,v}(\theta) = H, \text{ by (5.39)}.
\end{array}$$

¿From this, (5.9) and (5.33), recalling that $G_1(u) = B^H_u$, $G_2(v) = B^H_v$, we deduce that

$$G_3^{\kappa\varepsilon}(\kappa u) = a_{11}^{\kappa\varepsilon}(\kappa u, \kappa v)G_1(\kappa u) + a_{12}^{\kappa\varepsilon}(\kappa u,\kappa v)G_2(\kappa v) + Z_1^{\kappa\varepsilon}(\kappa u, \kappa v)$$

$$\stackrel{\text{(law)}}{=} a_{11}^{\varepsilon}(u,v)\kappa^H G_1(u) + a_{12}^{\varepsilon}(u,v)\kappa^H G_2(v) + \kappa^H Z_1^\varepsilon(u,v) \stackrel{\text{(law)}}{=} \kappa^H G_3^\varepsilon(u),$$

and in a similar way, $G_4^{\kappa\varepsilon}(\kappa v) = \kappa^H G_4^\varepsilon(v)$. Therefore (5.17) is proved. On the other hand, using (3.7) and (5.13), we obtain

$$9Q_1(\kappa u,\kappa v)Q_2(\kappa u,\kappa v) - \frac{9}{4}\lambda_{12}(\kappa u,\kappa v) = \frac{9}{4}[(\lambda_{11}(\kappa u,\kappa v)G_1(\kappa u) + \lambda_{12}(\kappa u,\kappa v)G_2(\kappa v))$$

$$\times(\lambda_{12}(\kappa u,\kappa v)G_1(\kappa u) + \lambda_{22}(\kappa u,\kappa v)G_2(\kappa v)) - \lambda_{12}(\kappa u,\kappa v)] \stackrel{\text{(law)}}{=}$$

$$\frac{9}{4}[\kappa^{-2H}(\lambda_{11}(u,v)G_1(u) + \lambda_{12}(u,v)G_2(v))(\lambda_{12}(u,v)G_1(u) + \lambda_{22}(u,v)G_2(v)) - \kappa^{-2H}\lambda_{12}(u,v)]$$

and consequently (5.18) is also proved. □

This achieves the proof of Lemma 5.1.

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